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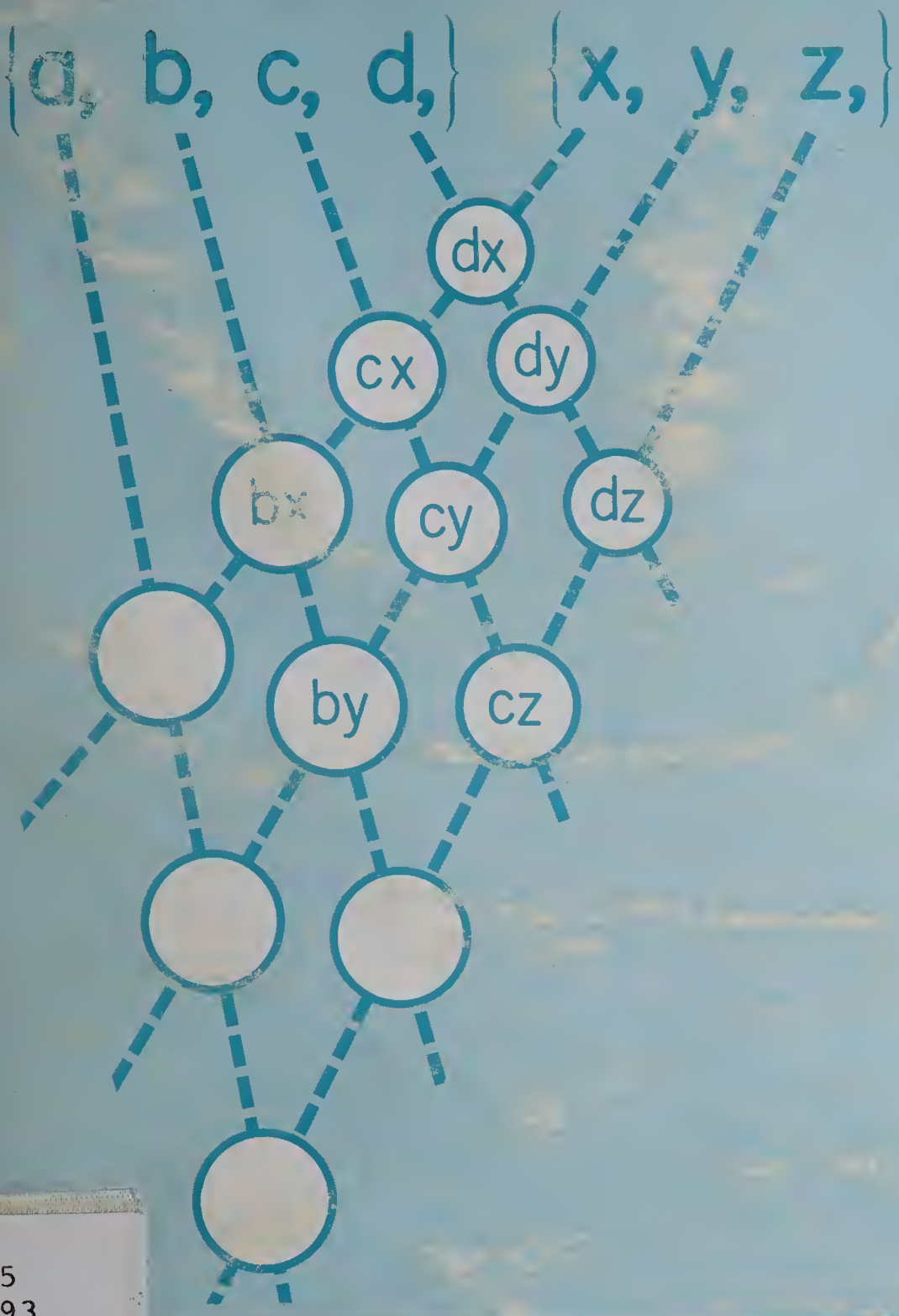

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MATH WORKSHOP

EVISED

Pre-publication Edition

Robert W. Wirtz ▪ Morton Botel

Max Beberman ▪ W. W. Sawyer

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Teacher's Guide

MATH WORKSHOP

REVISED

Pre-publication Edition

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PREFACE TO PRE-PUBLICATION EDITION

This edition of the Teacher's Guide contains excerpts from the Introduction to the regular edition and complete Notes on Pupil Pages for pupil pages 1 through 32. A separate booklet contains pupil pages 1 through 32, with answers.

Breaks in continuity in the Introduction due to excerpting are indicated by a series of dots across the page.

Circumstances beyond the control of the publisher have made this edition necessary. The regular edition will be delivered early in October.

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INTRODUCTION

I. A PERSPECTIVE

Loose talk about reform in the teaching of elementary mathematics has spread confusion and shattered the confidence of many teachers. Uneasiness and fear have developed. This mood has damaging effects in the classroom.

Teachers have been told they are teaching horse-and-buggy mathematics in a space age. New and esoteric language has begun to flood the literature and to be introduced into discussions. Some innovators have pointed the finger of scorn at traditional subject matter, pronouncing it "bad mathematics."

Relax! The mathematics our pupils are learning in traditional classrooms is good, sound, useful mathematics; it is good enough to play a central role in any moon shot. Traditional content in elementary school arithemtic is sound; it is at the core of man' s advance, including his recent burst into space.

Our real concern is with finding solutions for two interrelated problems. How can we help pupils (a) to learn more mathematics, and (b) to develop a deep appreciation for this increasingly important part of our culture? All proposals and discussions should be evaluated in terms of their contributions to the solution of these fundamental problems.

The traditional content taught to pupils in elementary schools is completely valid. Our elementary schools must teach pupils how to carry out the fundamental operations with whole numbers and fractions and how to use these skills in solving practical problems. But the truth of the matter is that pupils simply aren' t mastering the traditional content. Many pupils have too weak a grasp on basic facts such as $7 + 2 = 9$ and $8 \times 9 = 72$. (Most of these pupils will drop mathematics as soon as they can, losing contact with a tool that is essential in coping with the problems of the times.) The lack of mastery cannot be remedied just by giving more drill. Pupils will be able to master more of the traditional content if

we give them freedom to invent ways of finding answers, if we take steps to make sure that they understand what they are doing, and if they have ample opportunity to use what they learn.

The job of teaching more mathematics will become easier to the extent that pupils develop an appreciation for the subject. This appreciation can develop only when mathematics is learned with understanding and through active participation--it will not take place when mathematics is learned by rote. Mathematics is a world of endless fascination. It is a world brimming with surprises, patterns, and enticing vistas. To see a pattern emerge from a conglomeration of facts, to organize seemingly disconnected ideas--these are some of the joys of learning mathematics. It is this sense of excitement in mathematics which pupils ought to be experiencing in elementary schools.

Giant steps are currently being taken to alter the traditional approach to teaching mathematics in order to get more mathematics into the elementary and secondary curricula. To this end, brilliant pioneering work is being done at the Universities of Illinois, Maryland, Minnesota, Syracuse, Stanford, and elsewhere. Generous financial support has been given by the National Science Foundation, the Carnegie Corporation of New York, and the U. S. Office of Education for research projects, workshops, and institutes.

The authors of MATH WORKSHOP have participated in this pioneering work. Our program has evolved during a period of over ten years. Its ideas and techniques have been field-tested with thousands of teachers and tens of thousands of pupils.

II. CONTENT AND TEACHING APPROACH

The authors of MATH WORKSHOP believe that a sound mathematics program can be made ineffective by methods of teaching which rely mainly on rote learning. We also believe that even inspired teaching of isolated fragments of mathematics

will fail to produce pupils who understand and can use what they have learned. The MATH WORKSHOP puts its major subject matter emphasis on basic principles and unifying ideas which tie together the facts and manipulative procedures of arithmetic. This emphasis simplifies the subject, makes it understandable, insures recall, and increases interest. Our major pedagogical emphasis is that of teaching for discovery. We present content to pupils so that they are challenged to think about relationships, to look for patterns and clues, to draw logical conclusions for themselves, and to invent their own ways of finding answers.

A. The What of Mathematics

The Spiral Curriculum

The 24th Yearbook of the National Council of Teachers of Mathematics talks about "the basic mathematical understandings which should be continuously developed and extended throughout the entire mathematics curriculum, grades K-12 and beyond." This concept of the spiral curriculum has been used in designing MATH WORKSHOP. At each level of the MATH WORKSHOP the main ideas of mathematics have been introduced. Each successive year' s activities deal with these unifying ideas with increasing sophistication.

However, pupils are not expected to master such ideas as, say, multiplication of fractions or addition of positive and negative numbers the first time they meet them. They will encounter such ideas again and again, building always more layers of understanding.

The spiral curriculum rejects the "building block" approach to arithmetic which has been used in the past. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas with repeated reconsideration at later times. It calls for continual nurturing of important concepts year after year as understanding gradually develops.

The hope for absolute mastery must be limited to the basic facts of arithmetic. Here, complete mastery can and must be achieved. But where big ideas are involved, mastery does not come in this way. It develops slowly and is based completely on understanding.

The Unifying Ideas or Strands

MATH WORKSHOP emphasizes nine unifying ideas in mathematics:

1. Structure
2. Sets
3. Number and counting
4. Numeration
5. Addition and subtraction
6. Multiplication and division
7. Functions and relations
8. Geometry
9. Measurement

Each teacher' s guide page lists the unifying ideas considered on the corresponding pupil page and contains many suggestions for extending these ideas beyond what is called for on the pupil page.

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B. How to Teach Mathematics

It' s easy to ask a question so easy that almost anyone can answer it. And it' s easy to dig up trick questions--there are books full of them.

But challenging questions that encourage independent research and lead to discovery must be neither too hard nor too easy, and must not rely on clever tricks. Rather, challenging questions are less than obvious but will yield answers if the pupil applies thoughtfully what he already knows.

Such questions are hard to come by. They are worth their weight in gold. Often they lurk in very simple situations.

Practically every page and activity in MATH WORKSHOP poses such less than obvious, challenging questions. MATH WORKSHOP presents these questions through a wide assortment of techniques, many of which were invented by the authors of the program. Challenging questions are posed through the game of What' s My Rule?, Cross Number Puzzles, and through problem solving. Let us examine examples of these vehicles we use for asking challenging questions.

What' s My Rule?

In first grade, Miss Atkins tells the children that she has a secret rule and she will help them read her mind.

" Call out a number and I will answer with a number. "

" Five, " a child calls out.

" Four, " Miss Atkins answers. " Someone call out another number. "

" One, " a child calls out.

" Zero, " the teacher replies.

" Seven, " another child calls out.

" Six, " the teacher answers. " Have you guessed my rule?"

" Yes, " Bill is quick to reply.

" Don' t give my secret away. Just call out a number and tell me what you think I will reply. "

" If I say 9, you are going to say 8. "

" That' s right. I think that you know my secret. " Thus the children have learned a way to demonstrate their discovery.

Many children are not as quick as Bill. So Miss Atkins may put on the board a record of the development thus far, first listing the number called out and then the reply:

5 → 4

1 → 0

7 → 6

9 → 8

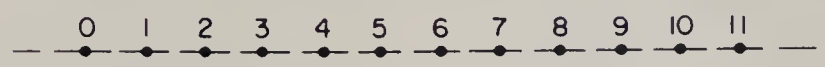
The last item was Bill' s contribution. Others might be:

6 → 5

2 → 1

4 → 3

When the game is new or the rule difficult, a number line on the board will help:



" Let' s act out the game we have played. When someone called out 5, I answered 4. Point to 5 on the number line. Keep your finger on that point. I answered by saying 4. Point to 4 on the line. "

Each step is similarly acted out.

" Now, if you call out 8, put your finger at that point on the number line. "

8 → ___

" What do you think I will reply?"

" Seven, of course! "

In some such way the idea can be made clear even to the slowest children. Care was taken not to talk about the teacher' s rule itself--only about the results of applying it. Thus, every child must make the discovery for himself--some with minimum clues, others only after several different kinds of clues. There is a bit of artistry in giving clues only when needed and finding ways to reserve at least a little for even the slowest child to find out for himself.

Usually some child will note a curious fact about this rule.

"Suppose that I call out zero. What will you answer?" a child may ask.

"Point to zero on the number line," the teacher suggests. "You can see that I would have to ask you to pick some other number." (Unless the teacher would like to talk about positive and negative numbers and zero.)

A new game begins with another secret rule. Some child may want to take over the teacher's role and invent a rule of his own.

Rules can become more complicated.

"I have a new rule. Call out a pair of numbers and I will reply by saying a number." Here is a record of a simple rule:

2, 3 → 5

4, 0 → 4

3, 1 → 4

1, 2 → 3

1, 1 → 2

3, 1 → ___

A more difficult rule is behind the following record:

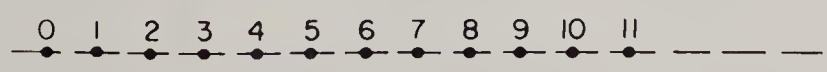
2, 4 → 3

0, 2 → 1

4, 0 → 2

5, 9 → 7

These last two items may puzzle the children. Perhaps a number line can help shed light on the rule.



"If someone called out 2 and 4, I would reply with 3. Point to the numbers called out (2 and 4). I would reply with 3.

"If someone called out 4 and 0, I would reply with 2. Point to 4 and 0 on the number line. I would call out 2.

"Point to 5 and 9--and remember that I would answer 7. What would I answer if someone called out 1 and 7?"

1, 7 → ___

"Four," Bill might answer, and quickly ask: "But what would you do if I called out 5 and 6 or if I called out 3 and 0?"

"I would have to ask you to pick another pair of numbers." (Unless the teacher is ready to discuss fractions.)

An unlimited number of rules can be devised on any level of sophistication. Further, there are countless ways to suggest different rules, such as:

a. 1, 4, 7, 10, 13, , , . . .

b.

3	1	4	10	0	9	6
7	3	9	21	1		

c.

7	4	3	1
9	9	0	9
11	6	5	1
	8	6	
6			2

Mathematics is essentially a study of patterns. It has its own fascinating patterns and it is often employed to help reveal patterns in the physical world. When a scientist or any observer is searching for a better understanding of what he notices, he usually tries to set down a description of events. Whenever possible he will try to write down numbers that in some way help him record relationships. He might record the following observations:

	1st	2nd	3rd	4th	5th
minutes	1	3	4	5	10
miles	1	9	16		

In his first three observations he notes time and distance. A pattern is strongly suggested. He may check his guess by making further checks for 5 minutes and 10 minutes. If his results are 25 miles and 100 miles, respectively, he will feel that he can formulate a rule or law that will suggest a pattern in certain real events. ($1 \times 1 = 1$, $3 \times 3 = 9$, $4 \times 4 = 16$, ...)

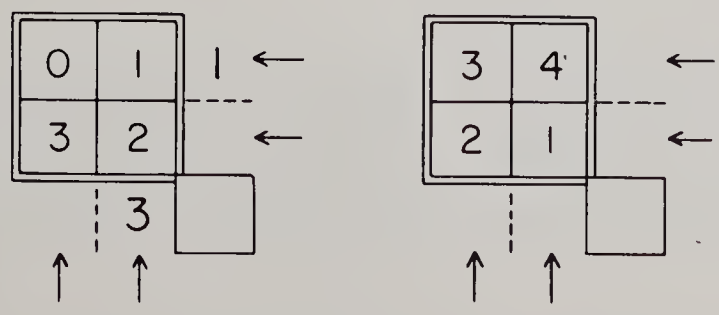
In mathematics we study patterns because they suggest basic principles of the subject itself and because mathematics helps record and explore patterns of the world we live in.

Thus, this exercise or game of What' s My Rule? has three important dimensions: (1) It can be used to motivate drill in the basic combinations of arithmetic. (2) It poses problems that children can think through to a solution. (3) It uses a method fundamental to the experimental sciences.

Cross Number Puzzles

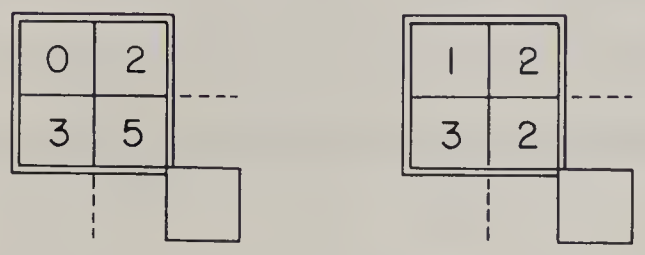
Too few of us realize that even six- and seven-year-olds can carry out research problems in arithmetic. Do you doubt it? Let' s leave speculation and get to work.

In the spaces indicated by arrows, write the sum of the two numbers listed in each row and in each column:



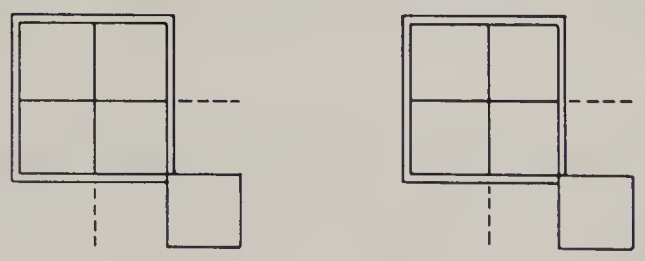
Do you have a desire to fill in the lower right-hand corner of each diagram above? Notice the two numbers listed above the corner and the two numbers listed to the left of the corner: 1 and 5, 3 and 3 in the first example; 7 and 3, 5 and 5 in the second example.

Most people would feel an urge to write 6 in the lower right-hand corner of the first example, and 10 in the corresponding corner of the second example. Try two more:



Again, a similar result. This is the kind of surprise that is the spice of arithmetic!

Can you select four numbers to list in the four blue-shaded cells so that some surprise happens in the lower right-hand corner? Try your hand at two examples:



Did you succeed?

This problem is relatively simple. Even first-graders can find examples that work. They are putting their arithmetic to use, practicing the basic facts of addition.

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Problem Solving and Applications

Story Problems. Over the years, "story problems" or "word problems" have been a source of vexation to many teachers of arithmetic, to many children--and to many parents. This activity, which properly seeks to relate the abstractions of mathematics to the world of events, often ends without a sense of accomplishment for anyone.

Story or word problems are usually reserved for the end of a development, as a sort of test or exercise of the pupil' s ability to apply an acquired knowledge of arithmetic.

Examples vs. problems. A critical analysis of the questions usually asked in traditional arithmetic programs reveals that they constitute primarily a test of a child' s reading ability, and that if the child can read, the questions are trivial and constitute examples rather than problems.

Example: Mr. Jones bought 8 fence posts. He paid $10 apiece. What was his total expense? (If you can read, this constitutes an example of the fact that $8 \times 10 = 80$.)

Problem: Mr. Jones bought 8 fence posts. He set them out in a straight line, 10 feet apart. What is the distance from the center of one end post to the center of the other end post? (If you can read, this is a problem that requires careful consideration of the situation presented; perhaps a sketch to show that there are 7 spaces between posts.)

The first situation is an example; the second is more of a problem. If a child clears the reading hurdle then the example taxes little more than his memory, while the problem depends on a less than obvious question that makes greater demands on the mind.

Examples and problems play different roles in helping children understand mathematics--and both are important.

When we notice a pattern repeating itself in the limited world around us--the world we can manipulate--we often use an expression in the language of arithmetic that points to that pattern, such as

$$2 + 3 = 5$$

At this point we ought to direct children' s attention to an application of this idea to the wide world of fact and fancy that they live in.

" Suppose that this is a headline. "

$$2 + 3 = 5$$

" Tell a story that might fit such a title or headline, " the teacher urges.

" There are five children in our family, two boys and three girls. "

" Fine. Who has another story that would fit?" the teacher asks.

" I have three cookies out of the five I started with. I ate two of them. "

" It' s the second of December and my birthday is on the fifth. My birthday is in three days. "

" On my hand I have a thumb and a little finger. There are three fingers in between. "

" Out of my allowance of five cents I have two cents left. "

" Three hours from now will be five o' clock. " (Check with a clock to determine whether or not this is a true sentence.)

Each of these situations has within it the conditions we point to in the headline. Each of the story-tellers reveals that he is beginning to grasp the relationship between arithmetic and the real world.

Whenever we abstract a bit of mathematics out of the world at hand and write it down in the language of arithmetic, good classroom technique should include the next logical step--to find examples from the world that reflect this idea.

The authors of MATH WORKSHOP believe strongly that basic mathematics should be derived from the world around us. For this reason we continually refer to practical situations on both the pupil pages and in the guide pages.

Each guide page contains a section entitled Headline for the Day. This might be copied on the chalkboard at the opening of the school day, thus giving pupils a chance to ponder it. For example, you might write the headline 2 + 3 = 5 given above, and then let pupils tell various stories. Or, you might put a story on the chalkboard and ask for headlines. For example, if you wrote the story:

Bill has 2 toy cards. His father gave him 3 more. You should expect headlines such as:

$2 + 3 = 5$

$5 - 3 = 2$

$3 > 2$

$5 = 3 + 2$

$5 - 2 = 3$

$5 > 2$

$2 + 3 = \square$

$\square - 3 = 2$

$2 < 5$

Please allow the pupils plenty of freedom in composing headlines.

We have included in the guide pages stories which you may wish to use on a drill sheet or put on the chalkboard. At least part of the discussion in each lesson should remind pupils that mankind has developed the language of mathematics to help report events that happen in the world of things.

Teaching for Discovery

Each page in MATH WORKSHOP and the activities suggested in the guide pages are designed so that there are several levels on which children can participate--

including opportunities for success for even the slowest learners.

The activities we present are truly open ended, with challenge for even the most advanced and able children in the class. While some questions will have been settled, more will have been left open. Some children will want to explore these open questions under their own steam. Such children ought to learn to work independently, sparing the teacher' s time for those who need it more. Actually, what the more able students need most is encouragement and opportunity to develop a taste for independent study.

The highest level of learning has been reached when a child begins to pose questions of his own and to work toward their solution.

There is in MATH WORKSHOP provision for introducing or practicing the basic facts of arithmetic. Usually such facts must be recalled and organized so that a pattern emerges. As patterns appear, they provide answers to less than obvious questions.

Discovering patterns is one of the most rewarding experiences in mathematics. It is called the "Aha! " or the "Gee-Whiz! " experience--a gasp; the light dawns! "I see it! " These are events teachers bend their best efforts to help bring about. One of the most disheartening facts teachers must face is that some children simply "don' t see it." Yet this is also a challenge--it calls for exercising skill in teaching. Slow learners need more clues--more obvious clues; but the finest teacher will reserve a wee bit for each child to "see" for himself. If the teacher' s best efforts fail, he may have to give away part of the secret and try again next time. This is an art that does not come easily, but the eventual payoff is deeply rewarding.

When the results of work in arithmetic exhibit an unusual regularity, the inquiring mind looks for an explanation. One problem may be solved but others are suggested. This is an exciting characteristic of mathematics.

When children discover a relationship, they are excited and eager to stay with that discovery. They will readily put this new idea to work, showing more examples of it. They come alive, often with an "Oo-o-o! I've got it!" or with a burst of industry using a newly-discovered relationship to reel off answers with speed and confidence.

What is the it they have glimpsed? Well, if we distract their attention with a request for them to tell about it or put it in words, we have selected a method that could not be better calculated to turn children against discovery.

That a child can move ahead in putting it to work successfully is certainly all the evidence we need in order to know that he has it. Our only justification for veering off in the direction of a word-pulling exercise is a feeling that it is a generalization that ought to be verbalized in order to be usable. But research does not support this as a valid reason.

Transfer to future experiences is not increased by attempting to verbalize a generalization. Subvocal or nonverbal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, is our main goal.

When children experience an insight--when they leap ahead because they have discovered a pattern--we ought to be very careful about letting words into the act too soon. The children probably do not have the language needed to express their discovery in words. If they try, we must, of course, help them. But too often such help and explanation degenerate into hairsplitting, an activity far removed from the act of discovery.

Gertrude Hendrix of the University of Illinois Committee on School Mathematics, who has done much research on discovery in the teaching of mathematics, warns that imprecise verbalization may "mutilate" discovery.

On the other hand, when some children (and teachers) make discoveries, they wish others to know about it. But herein lies a danger. Whenever a person reveals

his secret to one who has not been as fortunate, the latter is deprived of the delight of discovering it for himself.

So we invent ways in which children can share their discovery without revealing the secret too widely. We have them whisper their new idea in the teacher' s ear; we encourage them to make up additional examples which will provide further clues for those who haven' t found a pattern that underlies several examples. We name the new idea after its discoverer--Bill' s Rule or Mary' s Law or John' s Pattern. But we keep urging children to keep the secret--for a while at least.

Once in a while we find children who have not invented shortcuts for solving certain kinds of problems. For example, suppose that children use counting by one in counting problems which involve fairly large totals. Hopefully, such children will become discouraged and begin to look for easier ways of determining the total, such as counting by five. If the easier ways are discovered, the children should be highly praised and urged to demonstrate this technique to the entire class. If this example is not sufficient to shake all children loose from counting by one, then you must step in and urge the conservative ones to drop the slower technique. Please temper the discovery method with common sense.

III. REPRESENTATIONAL MATERIALS

When mathematics loses contact with the world of things and with the events out of which it came, the subject becomes mysterious to most children. If they do not understand that we are discussing the way things happen as we manipulate objects, they must resort to rote memory.

MATH WORKSHOP activities stress the fact that arithmetic provides a means of recording or reporting what happens in the physical world.

Many kinds of manipulative or representational materials are suggested and

discussed in this Teacher' s Guide. These include various types of counting materials, geometric shapes, Cuisenaire® rods,* number lines, etc.

Perhaps the most accessible source of representational materials is the pupil pages themselves. We believe that all of elementary mathematics can be abstracted out of experiences with the physical world. The very word WORKSHOP in the title of our program indicates our belief in the value of learning mathematics by working with physical objects. In some cases, the pupil pages are the workshop--pupils work with the two-dimensional pictorial representations of physical objects and apparatus (Cross Number Puzzles, number lines, colored strips) and use the forms provided to keep records. Frequently the teacher will decide that pupils need to work with three-dimensional equipment. In those cases, the teacher may provide equipment just like that pictured on the pupil pages or like that suggested in the guide pages.

IV. MASTERY OF BASIC FACTS AND ALGORITHMS

Mathematics provides a language to help report events that happen in the world of things. Counting is at its core. Because this process becomes tedious when many things must be counted, shortcuts to counting have been invented and improved. These shortcuts depend on two characteristics--memory and patterns. Thus, elementary school mathematics is essentially concerned with counting and with shortcuts to counting.

At first glance, MATH WORKSHOP activities may appear to be highly motivated exercises to provide practice in the basic facts and combinations and processes of arithmetic. In the vernacular, they wear the garb of "sugar-coated drill." Actually, they are valid educational devices, because they help establish that automatic recall of "fundamentals" without which children are mathematical stutterers.

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*Cuisenaire--Registered trademark of Cuisenaire Company of America, Inc.

Every teacher knows that many children are mathematical stutterers because they can' t easily recall the basic facts. They hesitate and flounder; the discussion bogs down; they lose the argument; they miss the feeling of movement toward a solution; they cannot enjoy mathematics.

This area of trouble is so universally known that it is surprising so many are only casually concerned! There is much hue and cry to bring new and unusual content into the elementary school curriculum--numeration in bases other than ten, sets, properties of numbers, and other concepts--yet there seems to be silence about such a crucial matter as helping children master the fundamental combinations without which mastery their progress in understanding mathematical ideas is very painful.

While we have emphasized the scope of our program in terms of nine unifying ideas, the basic facts of arithmetic and algorithms are taught and retaught throughout the program.

To provide you with help toward the goal of mastery, we have included on practically every guide page (in a section entitled DRILL) a sample work sheet which can be easily converted into a Ditto®* sheet. We have felt that it would be too expensive and perhaps a little silly to use any of the pupil' s pages for strictly drill purposes. (You may find it useful to send a Ditto drill sheet home with a pupil rather than have him take home his text.)

V. THREE WAYS TO USE PUPIL PAGES

There are at least three generally successful ways in which individual pages or groups of pages can be used in the classroom:

1) Use the page to conclude a development.

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*The name DITTO is a registered trademark of Ditto Incorporated, Chicago, Illinois.

2) Use the page as the basic for discussion.

3) Let children work independently.

1) In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching--with liberal use of manipulative material, the blackboard, and class discussion. The page is then used late in the development to tie the idea down, or simply to record the progress that has been made toward deeper understanding.

There are limitless ways to initiate lively activities around important and useful concepts. The MATH WORKSHOP provides many such activities for you to choose from.

After handling manipulative materials, playing number games, and other such preliminary work, pupils are ready to move through the page in the final few minutes of the arithmetic period.

2) In contrast, some pages might be gone through very carefully with an entire class. Such a page or sequence of pages offers many opportunities to dig deeply into: What' s going on here?

3) A third approach might be called Do-it-all-by-yourself. Begin by saying, "Open your book to (name the page). Look it over. Think about it. See if you can decide for yourself what is to be done. Wait a while before you ask for help. And let' s make today a ' quiet day. ' "

Often, the first question the page poses to children is: What am I supposed to do? There are almost always sufficient clues provided so that with a bit of study children can discover the intent of the exercises. Traditional materials have provided too little incentive for children to figure out for themselves the nature of the business at hand.

Now you feign reluctance to help but are, of course, watching carefully for those who haven' t sufficient self-confidence to get started on their own. If several

children are unable to get started, suggest that they work in pairs, helping each other quietly. Enlarge these helper combinations if necessary. Grouping like this will usually get the activity under way for everyone. Let time run out on the session if you wish, suggesting that those who haven' t finished may want to take the work home with them to finish.

VI. TEACHING AIDS IN PROVIDING FOR INDIVIDUAL DIFFERENCES

A. In the Teacher' s Guide

The Introduction presents a point of view and a rationale for our approach to arithmetic. You will find this useful in interpreting many of the things we have included in the pupil pages. You will also find it useful in explaining the program to parents.

Each pupil page is reproduced in the Teacher' s Guide in full size and in full color with answers printed in red.

Each pupil page is accompanied by guide pages which are divided into several sections. Let' s consider some of these sections:

Tasks. This section contains a very brief description of what the pupil will be doing as he works on the pupil page. We suggest that you study the pupil page in a pupil' s book and work on it before you read our description and the answers we give.

Purpose. Here we describe what we think the pupil will learn as he works on the pupil page or as he carries out some of the activities described in The Lesson. (Naturally, our description contains a conservative estimate of what might happen. Mathematics is full of surprises and so are pupils.) If you are aware of our purpose, you will find it easier to help pupils achieve it.

Unifying Ideas. In this section we list those of the nine unifying ideas to which the exercises on the pupil page and the recommendations in the guide can be expected to contribute. Please regard the list as a minimum statement, and be prepared

for pupils who may wish to go beyond what we have described for the page.

Materials for Teacher. This is a description of the equipment you would need if you followed all of the suggestions made in The Lesson. Some teachers will be able to look at this description and then plan their own lessons without reading further.

Materials for Pupils. This section describes the equipment needed by individual pupils in carrying out the tasks or in pursuing the activities suggested in the lesson.

The Lesson. In this section we describe one or more of the lessons we might teach after studying the pupil page.

Practically all of our lessons would involve representational materials. We might decide to use a workshop and laboratory approach, using the equipment directly suggested by the pupil page. In that case, the pupil page would be used as review material.

We might use equipment quite unlike that suggested by the pupil page, but in keeping with the purpose and unifying ideas. In some cases, the pupil page might be worked on at the beginning of the lesson, so that the laboratory would then provide an application of what was learned on the pupil page. In other cases, we might reverse the order and make the pupil page the application of what was learned in the laboratory.

We would assign some pupil pages to be worked on with very little explanation from us (see section V on page 18). For other pages, we would ask lots of questions. Hopefully, we would find some means of projecting such pages on the chalkboard so that we could talk about them easily (see section B, below).

No teacher could possibly find the time to use all our suggestions in any school year.

Choose the ones which best suit your style of teaching, and vary your choices from year to year.

B. "Math Workshop Supplementary Activities"

A package of supplementary activity worksheets is available at every level, A through F. Each package includes 32 copies of 64 different practice pages. These can be used independently by pupils for seatwork and for homework. Able pupils should be given supplementary activity material at advanced levels. Each set of supplementary activities includes a Teacher' s Manual with brief suggestions for each page and an answer key.

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G. "Games and Enrichment Activities"

We have prepared a manual on the use of representational materials, such as Cuisenaire rods and Geo-boards. These materials can be used as the basis for games and other enrichment activities--many of which are mentioned or introduced on the guide pages. They contain many unusual applications of the concepts considered in the pupil pages.

H. Tests in the Pupil Book

Scattered throughout MATH WORKSHOP, LEVEL C are pages which we suggest you use as tests. These are pages 22, 39, 52, 65, 88, and 99. The guide pages suggest point distributions for these tests.

As with any evaluation techniques for young children, care must be taken not to put more faith in tests than they deserve. Your own judgment of a child based on your face-to-face work with him is more valid and probably more reliable than scores on tests.

This brings up the general problem of marking or grouping pupils as they work on all of the pages of the text. The pages should be considered opportunities for investigation and exploration on the part of the pupil rather than as pages to mark for a grade. There are many pages on which pupils will not be expected to get 100% on the first try. Nothing would be more damaging to their regard for mathematical inquiry than to be pummelled with red marks and low grades. By all means, keep a light touch rather than a heavy hand when exploring these pages.

VII. SOME FINAL THOUGHTS

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B. Punctuation and Order of Operations

In LEVEL C the pupil will sometimes be faced with expressions such as:

$$8 - 2 + 3$$

which appear to be ambiguous. Does this expression stand for 9 or does it stand for 3? That is, should one subtract 2 from 8 and then add 3 to the result, or should one add 3 to 2 and subtract the result from 8?

One way to remove the ambiguity is to punctuate the expression with a pair of parentheses. Thus,

$$(8 - 2) + 3 = 6 + 3 = 9$$

and

$$8 - (2 + 3) = 8 - 5 = 3.$$

A second way to remove the ambiguity is to establish a convention or agreement. The convention we follow in MATH WORKSHOP is that in the case of an expression involving addition or subtraction operations and no parentheses, one should proceed from left to right in carrying out the indicated operations. Thus,

$$8 - 2 + 3 = 6 + 3 = 9$$

and

$$9 + 4 - 1 = 13 - 1 = 12$$

and

$$7 - 2 - 1 - 3 = 5 - 1 - 3 = 4 - 3 = 1.$$

In the case of an expression involving multiplication operations separated by addition or subtraction operations and no parentheses, the convention we follow says that one should perform the multiplications first, and then the additions or subtractions from left to right. Thus,

$$2 \times 3 - 1 = 6 - 1 = 5,$$

$$4 + 3 \times 2 = 4 + 6 = 10,$$

$$3 \times 5 + 7 \times 5 = 15 + 35 = 50,$$

and

$$4 \times 2 + 5 \times 3 - 6 \times 2 = 8 + 15 - 12 = 23 - 12 = 11.$$

• • • • • • • •

D. A Final Word

The MATH WORKSHOP has not invented a new mathematics. Nor have its authors spun novel theories of education. We have tried to bring into a whole the most universally accepted ideas of both educators and mathematicians.

Robert W. Wirtz

Morton Botel

Max Beberman

W. W. Sawyer

NOTES ON PUPIL PAGE 1

Tasks: Pupils count (in sequences suggested by the dotted lines), or use short-cuts to counting to fill in the blanks in a - n. Pupils then use this information to complete the number sentences in o - t.

Purpose: To review counting, addition (particularly with sums of 10 and of 5), number sentences, commutativity, and associativity.

Unifying Ideas: Structure; sets; number and counting; addition and subtraction.

Materials for Teacher: Demonstration-size tagboard or felt models of the various items pictured on pupil page 1 for flannel board, cork board, or chalkrail displays; three empty boxes; counters; Ditto masters.

Materials for Pupils: Cuisenaire rods or counters; Dittoed sheets.

The Lesson:

Counting the objects in a display amounts to matching, or pairing one-to-one, each word in the sequence

one, two, three, four, five, six, ...

with an object in the display. The last word used stands for the number of objects in the display. The first word in the sequence can be matched with any object in the display, the second word with any of the remaining objects, etc. No matter how the objects are chosen for matching, the counting procedure will lead to the same result. This is the major idea on pupil page 1.

You can prepare pupils for this page by using flannel board displays like those pictured on the pupil page.

Put ten counters of different shapes and sizes on a table or flannel board and have various pupils count them, each time changing the arrangement. Ask

whether the number of counters changed when the arrangement changed. Have pupils arrange ten counters at their desks in a random way and count them several times, each time changing the arrangement.

Have pupils place some of their ten counters on a sheet of paper and the rest off the sheet (perhaps a Ditto sheet marked for record keeping), keeping a record of how many such separations can be made. (They should find 11 separations.)

On	Off	Number Sentences
2	8	2 + 8 = 10
10	0	10 + 0 = 10

The Game of Three Boxes

Prepare three boxes with lids shaped like a triangle, a square, and a circle. Distribute ten objects among the boxes and let pupils count the objects in two of the boxes and guess the number of objects in the third. Let them keep a record:

10 = △ + □ + ○
10 = 3 + 2 + 5
10 = 4 + 3 + 3
10 = 8 + 1 + 1

This activity illustrates that a number (in this case, 10) can be partitioned in a variety of ways. When pupils understand that partitioning doesn't affect the total, they have an understanding of the mathematical notions of commutativity and associativity of addition.

Cuisenaire Rods

Let pupils make and compare trains of rods to illustrate such consequences of commutativity and associativity as:

$$2 + 3 = 3 + 2$$

$$4 + 10 = 10 + 4$$

$$(3 + 1) + 7 = 3 + (1 + 7)$$

$$(3 + 1) + 7 = (3 + 7) + 1$$

$$(4 + 2) + (3 + 5) = (3 + 5) + (4 + 2)$$

$$(4 + 2) + (3 + 5) = (4 + 3) + (2 + 5)$$

See pages 31-79 of the booklet Games and Enrichment Activities published by Encyclopaedia Britannica Press.

Make Up Stories from Number Sentences

Put on the chalkboard a mathematical sentence or "headline" such as: $9 + ____ = 14$. Let a pupil complete the headline: $9 + 5 = 14$. Then ask: Who can make up a story to fit this headline? The response may be slow at first if the pupils have not had previous experience with this game-like situation. They may begin with examples which are on the dull side, such as: There were 9 boys and 5 girls, or 14 altogether. Or, if a child prefers a question form, he might change the story to: There were 9 boys and some girls. There were 14 in all. How many were girls?

One way to help children break loose from parroting stereotyped word problems is to write several beginning sentences or phrases on the board as "thought starters." For example:

$$9 + ___ = 14$$

a. Harry and Jack captured some men from Mars...

b. Alice met the Mad-Hatter...

c. When the Pilgrims landed...

d. In the United Nations...

e. In two weeks there are...

f. Mary is very fond of...

g. Many satellites were circling...

h. I was playing hop-scotch...

i. On a baseball team, there are...

j. The telephone rang...

Drill

Consider using as a drill sheet page 1 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

You may also wish to include among drill activities some "translation" work in which a pupil is given a story or a description of a situation and is asked to write at least one mathematical sentence which could serve as a headline for the story. Here are some examples of stories and samples of the headlines which pupils might write:

1. Andy saved 5 pennies and 3 bottle tops.

$5 + 3 = \square$	$5 + 3 = 8$	$5 - 3 = 2$
$3 + 5 = \bigcirc$	$3 + \square = 5$	$5 - 3 = \bigcirc$
$5 > 3$	$3 < 5$	

2. Ned has 6 hamburger buns. Nellie has 4 hamburgers. Ned will give Nellie a bun for each hamburger.

$6 - 4 = 2$	$6 - 4 = \bigcirc$
$4 + \square = 6$	$4 < 6$

Stories such as these may be presented on Ditto sheets or written on the chalkboard. It is probably best to use the chalkboard presentation until pupils

become convinced that you really intend that they be creative in composing mathematical sentences as headlines for the stories.

Headline for the Day:

Write one story to fit all of the headlines O, P, S, T at the bottom of page 1.

NOTES ON PUPIL PAGE 2

Tasks: Pupils record the number of objects displayed in each pair of adjacent boxes. (Note that the four pairs of adjacent boxes at the top of the page are combined to form the 4-cell table at the left.)

Purpose: To review counting and addition.

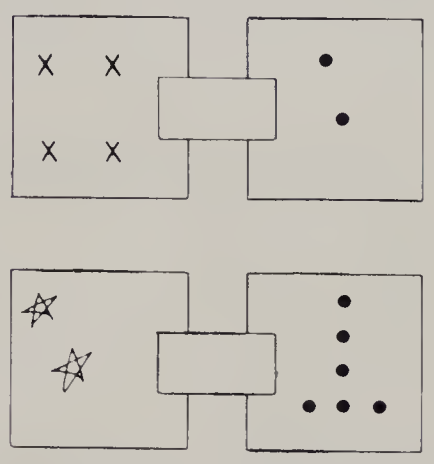
Unifying Ideas: Structure; number and counting; addition and subtraction.

Materials for Teacher: Flash cards (as described below); Ditto masters.

Materials for Pupils: Counters; Ditto copies of the sketches described below.

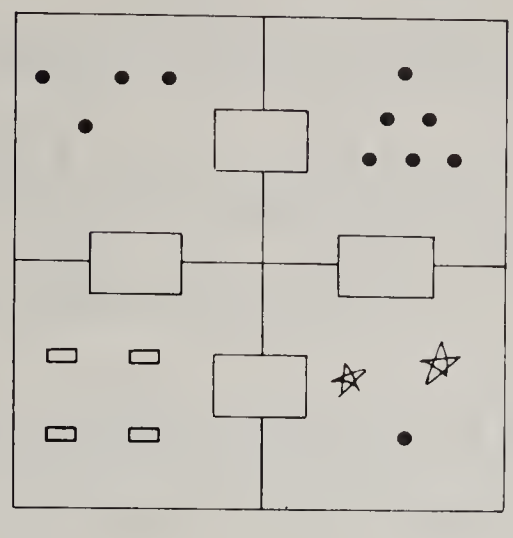
The Lesson:

This page provides another kind of counting, adding, and recording experience. You may wish to prepare pupils for this page in the following way. Draw the following sketches on the chalkboard and on Ditto sheets:

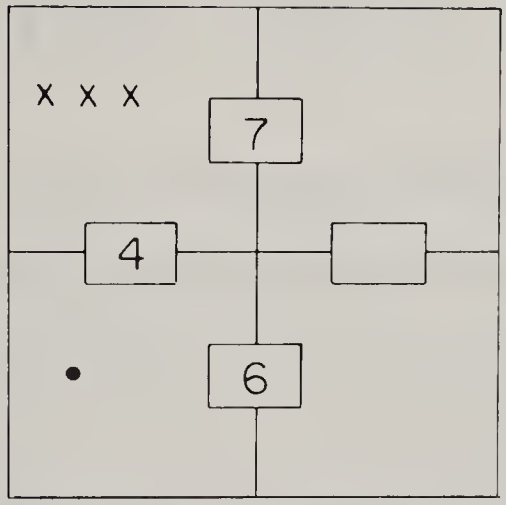


Point to the box with the ×' s and ask: How many ×' s? Then point to the adjacent box and ask: How many dots? Then ask: How many altogether? Four plus two is .... Write a 6 in the box in the middle. Pupils follow your lead by writing on their Ditto sheets. Now point to the middle box for the second pair of adjacent boxes (stars and dots), and offer the chalk to the class. Invite a pupil to come

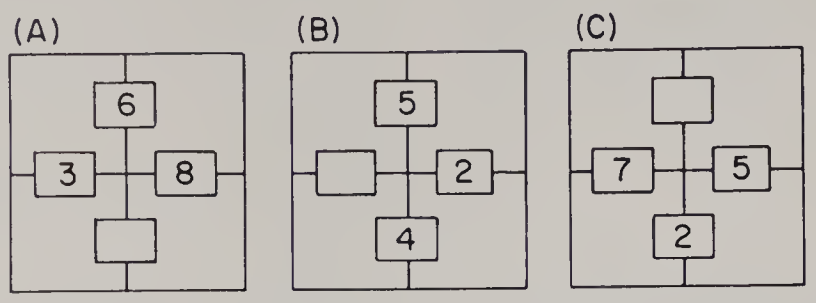
up and fill in the missing number. When they have grasped the idea, use more complex sketches like this one:



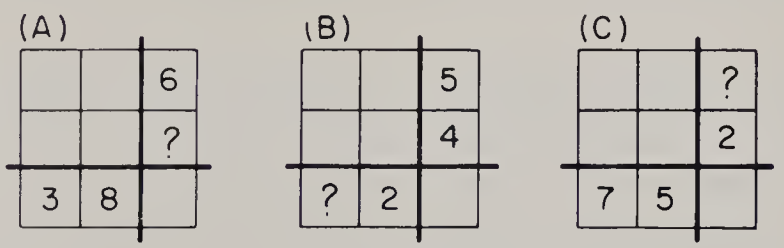
Now do reconstruction problems in which pictures are to be drawn and numbers recorded:



As a final touch, challenge pupils to fill in the missing numbers in sketches such as the following without drawing any pictures in the boxes:



These problems are equivalent to ones which require a pupil to find a missing row-sum or column-sum in a Cross Number Puzzle (see pages 9--10 of the Introduction):



The total number of objects displayed in the four boxes is the sum of the column-sums. It is also the sum of the row-sums. Hence, the missing number is easy to find. (Answers: (A) 5, because 6 + 5 = 3 + 8; (B) 7, because 7 + 2 = 5 + 4; (C) 10, because 10 + 2 = 7 + 5.)

Pupils who understand what is involved in the foregoing problems have a good grasp of the notion that you can change the order and the grouping in addition without changing the total.

Flash Card Activities

We describe below flash card activities which are useful for diagnostic and remedial work.

Flash cards are a time-tested device to help develop the ability to recall

without hesitation the hundred basic facts of addition and the hundred basic facts of subtraction.

The use of flash cards as the only device to accomplish this job is not wise. Their use as punishment is almost always self-defeating - a resisting mind is not a receptive mind.

Flash cards are most effectively used in a gamelike atmosphere. In a race against time the learner competes with himself.

A valuable use of flash cards is a diagnostic one. They can be used to locate the particular facts that are most troublesome. Therefore, the set of cards used ought to include all 200 facts.

Flash cards may be made with 3" by 5" cards. On one side of a card the fact is indicated by an open sentence: $5 + 3 = \square$. On the back of the same card, the complete statement is given: $5 + 3 = 8$.

In preparing a set, you may find the following check list helpful:

Second Number

First Number	0	1	2	3	4	5	6	7	8	9
0	a✓	b✓	c✓	d✓						
1										
2										
3										
4										
5										
6										
7										
8										
9										

a✓ $0 + 0 =$ ___

b✓ $0 + 1 =$ ___

c✓ $0 + 2 =$ ___

d✓ $0 + 3 =$ ___

Each cell points out one of the basic facts of addition. To come up with the total list of facts, simply complete the table, moving across the cells horizontally.

For each addition fact, such as: $3 + 5 = 8$, there is a subtraction fact: $8 - 5 = 3$.

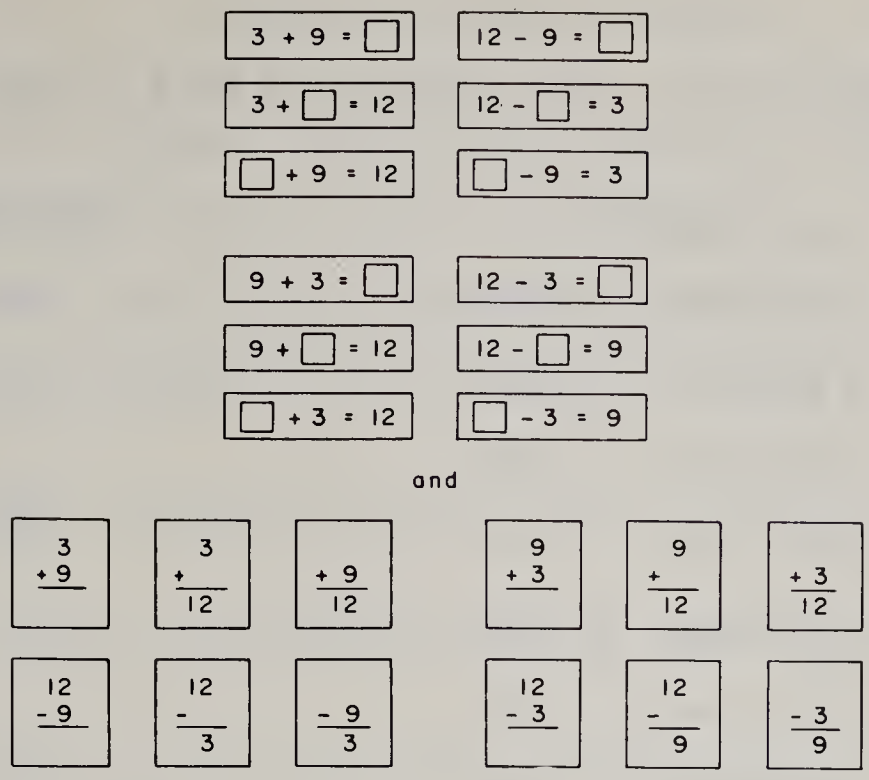
Used diagnostically, an appropriate group of flash cards (consisting of facts children have already met and considered) are mixed up thoroughly - and the game begins.

As the child responds, separate the cards into two piles - those he answers without hesitation and those he does not call up automatically.

The second group delimits the child's problem. Separate this group into two groups as he goes through them again - those on which he only hesitates and those which he gets wrong or must "figure out." On the former, the child needs a little practice; on the others he needs to work hard. To help him master the latter group, follow these procedures:

(a) For each fact, be sure that he draws a diagram or manipulates counters, and then reasons to a correct conclusion.

(b) Make supplementary cards for each of these facts, using as many different forms of the fact as you have time to make. Suppose that he has trouble with: $3 + 9 = \square$. The following forms are recommended:



This great variety of arrangements causes the child to think through the question in many related ways and thus reinforces his memory. Furthermore, the variety helps to deepen his understanding of important relationships among the facts of arithmetic. By using many forms of a question you help keep the practice lively instead of allowing it to become dull and monotonous. From time to time that group of the 200 basic facts with which the child is having trouble ought to be presented in diagnostic fashion, working each time to eliminate more and more that have already been committed to memory.

Important: Remember that this drill is most effective when it is interspersed often with quite dissimilar activities and when a gamelike aspect is maintained.

Drill

Here is a suggestion for some simple drill work on place value:

	long way	short way
0.	3 tens and 4 ones	34
1.	2 tens and 3 ones	____
2.	5 tens and 2 ones	____
3.	__ tens and ... ones	38
4.	1 ten and 5 ones	____
5.	6 tens and 6 ones	____
6.	2 tens and 0 ones	____
7.	__ tens and ... ones	50
8.	__ tens and ... ones	98
9.	__ tens and ... ones	89
10.	8 tens and 1 one	____

Here are some stories for which pupils can be asked to write one or more headlines:

1. Joe ate 3 of his 11 cookies.
2. Lyle had 385 stamps. He put 153 of them in a stamp book.
3. There were 6 black horses and 7 white horses at the rodeo.

Headline for the Day:

Jody bought some blue pencils and some green pencils. He bought 12 pencils altogether. What can you say about Jody's purchase? ..... Make a table.

NOTES ON PUPIL PAGE 3

Tasks: Pupils use the "jump" clues provided to label the required landing points on the number lines in A-D. For the exercises in the lower half of the page, pupils complete each counting sequence in accordance with the indicated path. Note that A corresponds with E-I, B with J, C with K, and D with L.

Purpose: To review counting by 1's, 2's, 3's, 5's, 10's and 25's.

Unifying Ideas: Number and counting; functions and relations.

Materials for Teacher: Two empty boxes; small objects; masking tape (for number line chunks); Ditto masters; colored chalk.

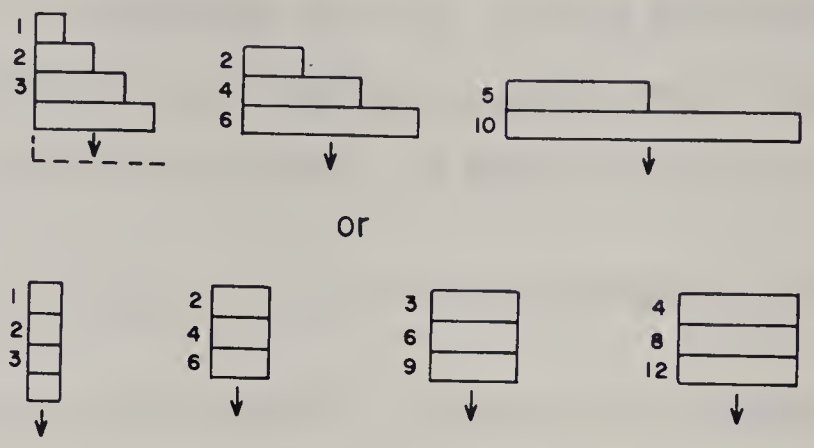
Materials for Pupils: Cuisenaire rods; Ditto sheets.

The Lesson:

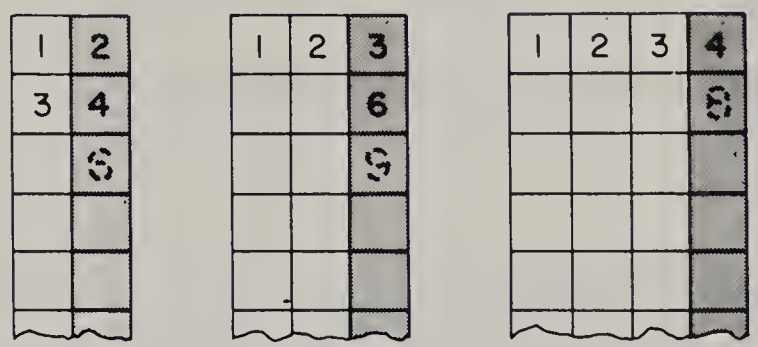
Throughout MATH WORKSHOP you will note a recurring emphasis on pattern. Pupils explore situations in which items are connected to each other logically. They will find the beginnings of patterns and their job is to use the given clues to extend or complete the patterns. Quite frequently such pages can be initiated by doing "silent arithmetic" (see the booklet Games and Enrichment Activities, pages 1-5 and 9-11) while you are working at the board. At other times pupils may be encouraged to try to complete the page on their own with no instructions from you. Perhaps pupil page 3 is a page you may wish to ask your pupils to try without instructions.

You may prefer to introduce the page (or follow it up) with any of these activities:

1. Using Cuisenaire Rods or measured strips, have pupils build models and extend the following patterns:



2. Have pupils number shaded areas only:



3. Have pupils color in every second (third, fourth, etc.) number block to form a pattern. The example shown below is for 3. Such colorings make interesting "wallpaper" patterns.

1	2	**3**	4	5	**6**	7	8	**9**	10
11	**12**	13	14	**15**	16	17	**18**	19	20
21	22	23	**24**	25	26	**27**	28	29	**30**

Number Blocks Colored

Another good way to introduce this page and almost any other in the book is to project a transparency of the page on the chalkboard or a screen.

Number Lines

In designing pupil page 3, we have assumed that pupils are already familiar with number lines from their work in an earlier grade. If this is not the case for your pupils, you will need to carry out some preliminary work. Use a cardboard strip to mark the 0-point and the 1-point, and use two cardboard strips (of the same length) to mark the 2-point. It is essential that pupils understand the significance of the 2-point - the distance between the 0-point and the 2-point represents the length of a 2-car train, etc. Make up some exercises on a Ditto sheet which are like those on pupil page 3.

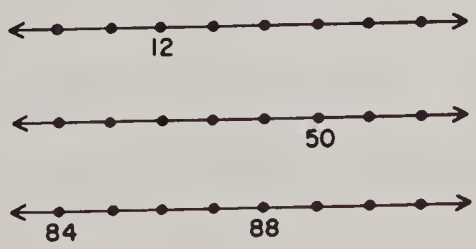
Play some games of counting forward and counting backward similar to the following:

From the number 10 count backward two numbers; then count forward three numbers.

From 60, count backward five numbers; then forward five numbers.

What are the numbers between 30 and 40?

Draw some number line chunks on the board (or make them out of masking tape) similar to the following and discuss naming the points:

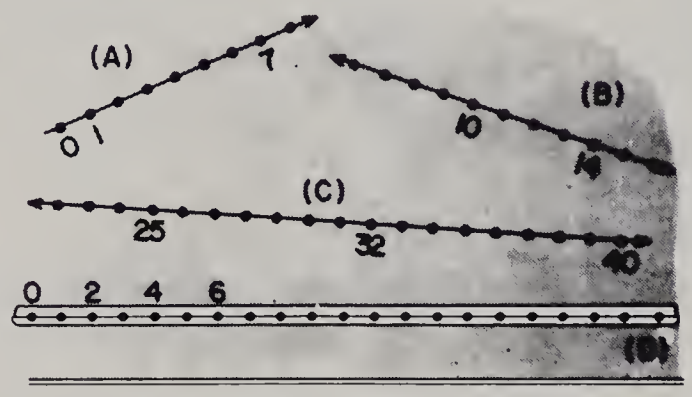


For some extremely simple number line activities, see pages 1-4 and page 13 of the booklet Games and Enrichment Activities. Be sure to modify the activities suggested there by including a marked point for 0 on each number line drawn.

When the foregoing preliminary activities have been completed (or in case your pupils are already familiar with number lines), you may wish to carry out the following additional preparatory activities.

Activity 1

Display several number line chunks on the chalkboard; A, B, and C in chalk, and D on masking tape:



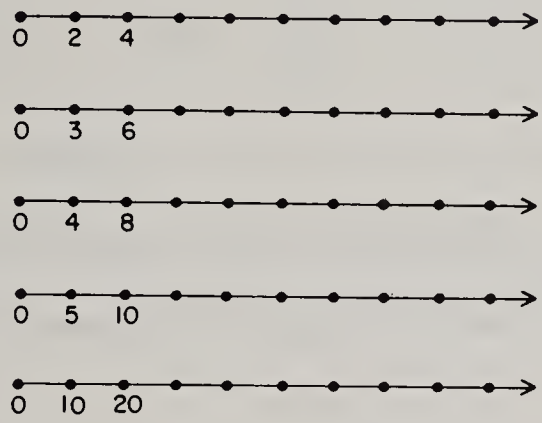
Select volunteers to complete the labeling of number lines A, B, and C. For number line D proceed as follows. Start at the 0-point and label consecutively the first few multiples of 2: 0, 2, 4, and 6. Stop and offer the red chalk to a volunteer who should label the next few multiples of 2. Have him offer the red chalk to another classmate to continue the labeling a few more steps, etc. What numbers did we use? (The even numbers.) Now, using green chalk, label points 1, 3, and 5 in that order. Stop and offer the green chalk to a volunteer to label the next of the unlabeled points. Have him pass the green chalk to another

classmate to continue the game. What numbers did we use this time? (The odd numbers.)

Erase the board. Who remembers all the numbers we wrote in red chalk? List them on your paper. Billy, read your list. Make a list of the numbers we wrote in green. Jane, read your list to the class.

Activity 2

Distribute Ditto sheets showing the following and ask pupils to label the marked points:



Activity 3

Draw a number line on the chalkboard, label the points for the even numbers, and show the connection between a jump and an addition sentence. Start with jumps of 2 or 4 and with even numbers as starting points. For example, draw a jump from 4 to 6 and write the sentence: $4 + 2 = 6$. Then write the sentence: $6 + 4 = 10$. Ask a pupil to draw the jump. (Encircle the labels for the landing points.) Change to jumps from odd-numbered points to even-numbered points and from odd to odd.

Activity 4

Draw a number line on the chalkboard and present the compound sentence:

$9 + 5 = \square$ and $\bigcirc + 9 = 14$

To find the missing sum, one starts at 9, jumps 5 to the right, and lands on 14. Then, to find the missing addend (in the second part of the sentence), one starts at 14 and jumps 9 to the left. A pupil who is aware of the fact that addition is commutative should be able to predict the landing point for the second jump. Continue with other compound sentences such as:

$3 + 7 = \square$ and $\bigcirc + 3 = 10$

$4 + 11 = \square$ and $\bigcirc + 4 = 15$

$13 + 5 = \square$ and $5 + \bigcirc = 18$

How Many Ways?

Use two boxes - one red and one blue - and fourteen objects.

In how many ways can we arrange all fourteen objects in the boxes so that in each case there are at least ten objects in one of the boxes?

The condition that there must be at least ten objects in one of the boxes must be kept in mind.

As the different arrangements are made, a record is kept and appropriate mathematical sentences are noted:

	Red	Blue	
Mary	10	4	10 + 4 = 14
Bill	2	12	2 + 12 = 14
Jane	12	2	12 + 2 = 14
Al	0	14	0 + 14 = 14
Fred	4	10	4 + 10 = 14
Sue	13	1	13 + 1 = 14
Tom	14	0	14 + 0 = 14
Jim	11	3	11 + 3 = 14
Gene	1	13	1 + 13 = 14
Sarah	3	11	3 + 11 = 14

Is that all? Can you suggest a plan for making the arrangements that would help us be sure that we haven' t missed a possibility?

Can there be the same number of objects in each box under the conditions of this problem?

Can one box have 10 more objects than the other? (Yes)

... 1 more? (No) ... 2 more? (No) ... 3 more? (No) ... 4 more? (No)

... 5 more? (No) ... 6 more? (Yes) ... 7 more? (No) ... 8 more? (Yes)

... 9 more? (No) ... 11 more? (No) ... 12 more? (Yes) ... 13 more? (No)

... 14 more? (Yes)

Drill

Consider using as a drill sheet page 2 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for which pupils can be asked to write one or more headlines:

1. Tami had 6 cents. She spent 3 cents.
2. Sam sold 10 of his 17 newspapers.
3. There were 56 cookies baked for a party at Fran' s house. Her guests ate 37 of these cookies.

Headline for the Day:

Connie took some jumps of the same size starting from 0 on a number line. She stopped jumping when she landed on 36. What can you say about the size of each jump Connie took?

NOTES ON PUPIL PAGE 4

Tasks: Pupils use the clues provided to continue each sequence as required.

Purpose: To review counting by 1's, 2's, 3's, 4's, and 5's.

Unifying Ideas: Number and counting; functions and relations.

Materials for Teacher: Masking tape for number line chunks.

Materials for Pupils: Cuisenaire rods or counters if needed.

The Lesson:

There are three aspects of the activities on this page which bear comment.

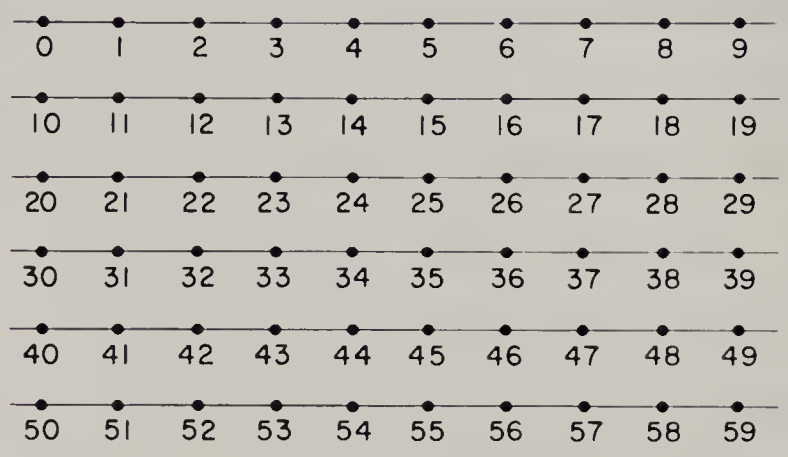
1. In rows b, c, and d the number patterns above and below the lines are related and pupils should be helped to observe this relationship through such questions as: In line b, do the 2-jumps and 3-jumps ever land on the same points? (Yes: 6, 12). If we continued the line, where would they land together next? (18). After that? (24). Record the points: 6, 12, 18, 24, ___.

2. In lines c and d the curved arrows suggest patterns. These patterns are easily discovered if we think of a curved arrow as representing the words "goes to." Thus in c 0 goes to 1, 4 goes to 5, 8 goes to __, etc. What would 20 go to 24? 28? etc. In d 5 goes to 3, 10 goes to 8, etc. What would 25 go to? 30? 35? etc.

A table listing these moves is helpful in trying to discover the relationship. Pupils should be asked to extend the pattern another five to ten steps. Use Cuisenaire rods or counters to help pupils who don't discover these relationships even after studying the tabulations.

c		d	
0	→ 1	5	→ 3
4	→ 5	10	→ 8
8	→ 9	15	→ 13
12	→ 13	20	→ 18
__	→ ...	__	→ ...
__	→ ...	__	→ ...

3. In exercises e to i we have increasing sequences and decreasing sequences. If pupils need help with them, use number line chunks drawn on the board or on a permanent chart. Have pupils use their fingers or chalk to demonstrate the patterned jumps.



When the page is completed, ask: How are rows e and f related? Does the pattern in row g look familiar? Does anyone know the special names for the numbers in rows e and i? (even numbers; odd numbers).

Making Up Stories from Number Sentences

In MATH WORKSHOP we encourage pupils to write or tell real or fanciful stories that have in them the conditions suggested by number sentences. The numbers themselves often generate ideas. Put several open mathematical sentences on the board:

a. $12 - ___ = 8$ b. $7 + 7 = ___$

c. $___ = 25 - 10$ d. $100 + 25 = ___$

e. $___ = 11 + 11$ f. $___ + 100 = 300$

Say to the pupils: First, let' s complete these sentences, and then we shall select one as an appropriate headline for a story.

a. $12 - 4 = 8$ b. $7 + 7 = 14$

c. $15 = 25 - 10$ d. $100 + 25 = 125$

e. $22 = 11 + 11$ f. $200 + 100 = 300$

Now make up a story about one of the headlines.

The relation between the facts of arithmetic and story problems is not obvious to children. They need help in determining the conditions in a series of words that can be expressed in a mathematical sentence or formula.

As children exercise their ability to make up stories to fit headlines, they begin to understand what goes on in the minds of others who make up story problems. The mysteries disappear. Once they can make up stories, children gain confidence that they can reverse the process. They begin to understand the relation between life situations and mathematical sentences. Teachers who have used this approach find that pupils become enthusiastic about being allowed to express a creativeness they never associated with arithmetic.

Very often, word problems present a difficulty because the child' s reading ability is not yet sufficiently developed. He struggles with reading the words; he is unsure; he is puzzled. In such a state of mind, he is not prepared to extract a mathematical problem from words that are already a reading problem. The method outlined above — from headlines to stories — shows the connection between a mathematical idea expressed in arithmetical symbols and a life situation expressed in words. A pupil need not read at all to gain confidence in

this area. Then, when he must go the other way — from stories to mathematical sentences — he is better equipped to find the cues he needs. He has tackled two very different problems but he has tackled them one at a time.

Drill

Here is a suggestion for a drill sheet:

Watch the Signs!

3	6	5	5	□	○
+1	− 5	+ □	− ○	+ 5	− 2
○	○	7	3	6	2

2 + 3 = □ 3 + □ = 7 ○ + 8 = 8

6 − 2 = □ 5 − ○ = 2 □ − 1 = 4

3	□	5	□	2	4
−○	− 5	+ 3	+ 6	+ ○	− 3
1	0	○	10	4	□

6 − ○ = 2 □ + 6 = 9 4 + ○ = 8

□ − 2 = 5 7 + 5 = ○ 4 − 1 = □

7	6	○	7	6	5
+ ○	− □	− 0	+ 8	− ○	+ 7
10	3	8	□	6	□

□ + 4 = 13 9 − ○ = 7 7 − 4 = ○

4 − ○ = 1 8 + □ = 12 □ + 9 = 11

□	3	○	10	3	2
− 5	− ○	+ 9	− □	+ 8	+ □
4	3	13	4	○	9

Here are some stories for which pupils can be asked to write one or more headlines:

1. Ted had 57 cents. He spent 50 cents.
2. Lester has 83 blue marbles and 75 green marbles.
3. Maria saw 6 horses with silver saddles and 10 horses with brown leather saddles.

Headline for the Day:

Add the numbers listed in exercises e on page 4 to the corresponding numbers in exercise i. What can you say about the sums of these numbers?

NOTES ON PUPIL PAGE 5

Tasks: Pupils use the clues provided in the boxes A-F or in the chart below the boxes to record the required entries. For the What are My Rules? games, pupils extend each sequence as required after discovering the rules of formation. For V at the bottom of the page, pupils use the clues provided by the first, third, and sixth columns (or the pattern indicators at the extreme right of the table) to record the appropriate entries.

Purpose: To review concepts of addition and subtraction.

Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations.

Materials for Teacher: Number line chunks; Ditto masters.

Materials for Pupils: Ditto sheets; Cuisenaire rods for activities described in pages 38-64 of the booklet Games and Enrichment Activities.

The Lesson:

The objective of the top half of pupil page 5 is for pupils to see the relationship between the display of letters in boxes A-F and the entries on the chart below the boxes. Looked at in a certain way, each display of letters represents a subtraction situation. For example, the display in box A is a representation of the fact that $8 - 3 = 5$. Obviously, recognizing the words has nothing to do with the problem. The record is concerned with counting the total number of letters, the ones crossed out, and the ones not crossed out.

Of course, pupils should be helped to see, before this lesson is completed, that these same pictures can be interpreted as suggesting other members of a fact team. In A, for example, all of the following fit:

$3 + 5 = 8$ $\qquad$ $8 - 3 = 5$

$5 + 3 = 8$ $\qquad$ $8 - 5 = 3$

So do the "reverse" statements:

$8 = 3 + 5$ $\qquad$ $5 = 8 - 3$

$8 = 5 + 3$ $\qquad$ $3 = 8 - 5$

Chalkboard Activity 1

If you find that pupils are not able to go directly to the page and work on their own, you may wish to conduct some class activities at the chalkboard similar to those on pupil page 5, but using the idea of addition:

I

He is
here

II

Bill ran
fast

III

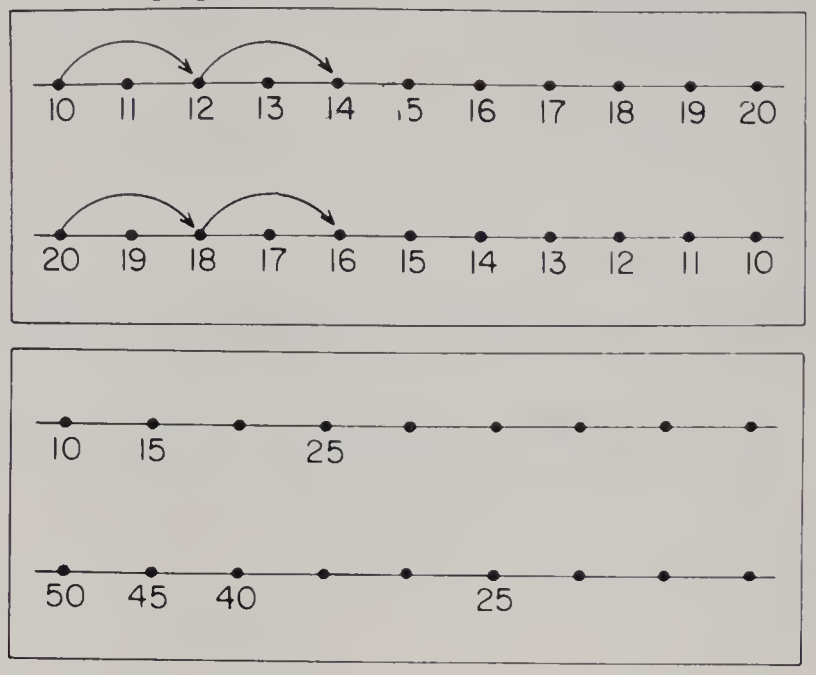
IV

	I	II	III	IV	V	VI
Letters above the line...	4					
Letters below the line...	4		5	0		
Total letters...		11	8			

Pupils can include their own words in III and in IV from the clues given, and in V and VI they can either draw additional boxes or imagine the words and boxes.

Chalkboard Activity 2

Draw number line segments and begin jumps or labeling as in the examples below. Offer the chalk to pupils to continue the patterns.



Chalkboard Activity 3

2	8	5	2			10			a
7	4	3	9	20	0				b
9	12			23	19	20			a + b

The letter labels on the right are to be regarded as pattern indicators (or row names) and no special attention need be called to them unless pupils wish to discuss them.

You may wish to follow up this page with further, similar examples which you can quickly prepare on Ditto masters, or with such exercises as the following:

Enrichment Activity 1

Put the missing operation sign in the shaded block to make the sentence into a true one.

2 ☐ 3 = 5	10 = 5 ☐ 5
5 ☐ 3 = 2	5 = 10 ☐ 5
5 = 3 ☐ 2	4 ☐ 3 = 7
4 = 2 ☐ 2	7 = 3 ☐ 4
4 ☐ 2 = 2	3 = 7 ☐ 4
8 ☐ 3 = 5	7 ☐ 3 = 4
8 = 5 ☐ 3	6 ☐ 1 = 5
3 ☐ 5 = 8	5 ☐ 1 = 6
3 = 8 ☐ 5	1 = 6 ☐ 5

Enrichment Activity 2

Tell whether the sentences are true or false.

	T	F		T	F
$1 > 2$		F	$1 + 1 = 2$		
$2 = 1$			$2 = 3 - 1$		
$2 = 2$			$1 + 1 > 2$		
$4 = 4$			$2 < 1 + 2$		
$4 > 3$			$4 = 2 + 2$		
$3 < 4$			$4 > 2 + 3$		
$6 = 6$			$4 = 2 + 3$		
$6 > 7$			$2 + 3 > 4$		
$7 > 7$			$3 - 2 = 5$		
$8 > 7$			$6 = 3 - 3$		

Enrichment Activity 3

Pairs of Numbers is an activity that requires thinking, and does not involve large numbers. A notation that may be new for many children is used in asking a question that involves pairs of numbers. For example:

This is an open sentence. The square and triangle are to be considered as "reserved seats" for pairs of whole numbers. If 1 approaches the square, and if 1's partner is 3, and they both sit down, they make the sentence into a true one because 1 + 3 is 4. Another pair that would make the sentence into a true one is (0, 4). A record of pairs with sum 4 can be made as follows:

	A	B	C	D	E	F
□	1	0	3			
△	3	4	1			

The pairs mentioned above have been recorded under A and B in the chart. Under C another pair has been recorded. Are there more? Yes, 4 for the square and 0 for the triangle; also, the pair (2, 2). Five pairs, so far. Are there more pairs of whole numbers which can make the sentence into a true one? No!

This activity does not involve much computation but the questions it raises are not trivial. Pupils are given an opportunity to record the suitable pairs in a pattern that make it easy to answer the question: Are there any more pairs of whole numbers that can make the sentence into a true one?

0 and 4
1 and 3
2 and 2
3 and 1
4 and 0

The following open sentence may look like the previous one but the results are very different:

$\square - \triangle = 4$

Pairs of whole numbers which satisfy this sentence — that is, which turn it into a true sentence — include (4, 0), (10, 6), and (5, 1). Are there more? Yes. And if the pairs were to be arranged in a pattern, it would become clear that it is possible to find an unlimited number of pairs for these reserved seats which satisfy the open sentence.

4 and 0

5 and 1

6 and 2

7 and 3

8 and 4

9 and 5

etc.

After pupils have become proficient at finding pairs of numbers which satisfy open sentences such as: $\square = \triangle + 3$, $\square + \bigcirc + 1 = 12$, $\square - 2 > \triangle$ introduce a variation by giving them an open sentence that contains three "holes" or reserved seats, two of which are of the same shape and size. For example:

$\triangle + \square + \square = 15$

The rule here is that whatever number sits in one of the squares must also sit in the other square. If 1 sits in the triangle and 7 sits in each square, the sentence is turned into a true one. We say that the pair of whole numbers (1, 7) satisfies the open sentence. Are there any more pairs of whole numbers that satisfy this open sentence? Yes; here is a complete list:

△, □

1, 7

3, 6

5, 5

7, 4

9, 3

11, 2

13, 1

Note well that the rule which says that frames of the same shape and size must hold places for the same number does not imply that frames of different shape or size must hold places for different numbers. In the example just considered, one of the pairs of numbers that satisfy the open sentence is (5, 5). Also, in the very first example, we found that the pair (2, 2) satisfied the open sentence.

From time to time we shall make further suggestions for Pairs of Numbers activities.

Enrichment Activity 4

Much practice on the basic facts of arithmetic can be concentrated in a small space in exercises like the following:

0		1		
1				
2		3		
3	3			
+	0	1	2	3

8				
6				
4				
2				
+	2	4	6	8

		10		
			10	
				10
	2			
+	1	3	5	7

Change the orientation in the grid:

+			4
	4		8
			6
	0	2	

+	7		
8		17	19
		19	
	19		

	17	16	+
19			
20		18	
10			

Change the operation in the grid:

15	10		
10	5		
5	0		
−	5	10	15

−			

			−

What Are My Rules?

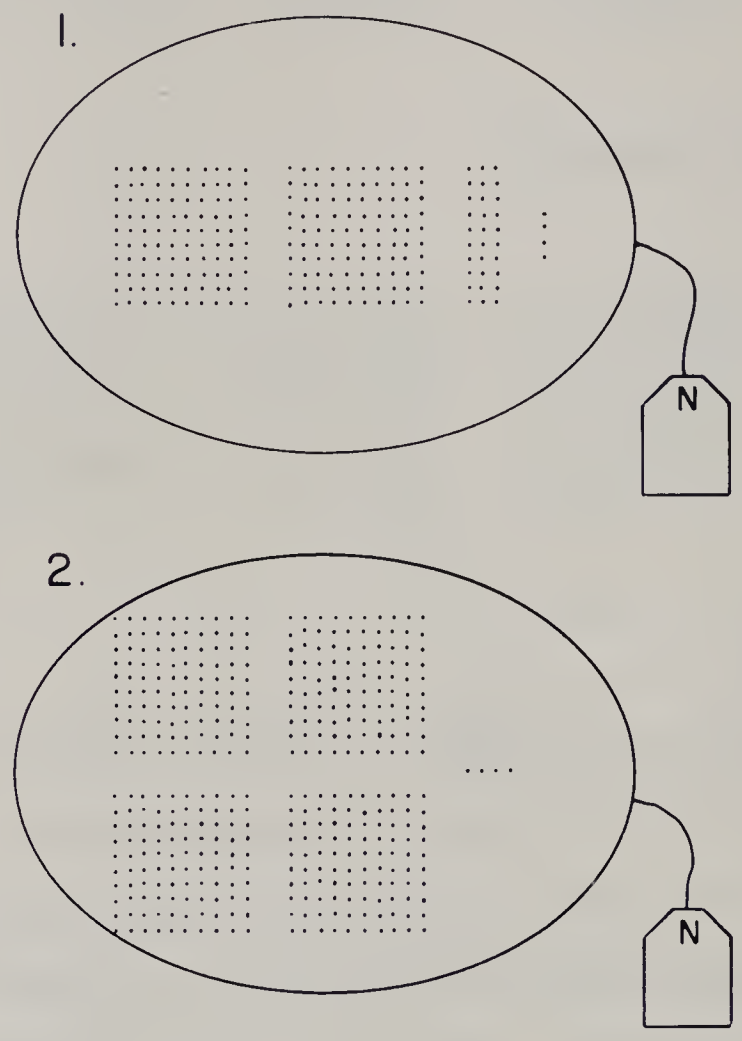
The game of What Are My Rules? is played frequently in MATH WORKSHOP. (See page 5 - 9 of the Introduction.) It is fundamentally a discovery game in which pupils search for patterns and extend the patterns according to discovered rules. We use this game whenever we wish to tell pupils to do something without having to explain in words. What Are My Rules? tells them to look for clues in the arrangement of pictures and numbers which indicate what needs to be done. This requires time for reflection, thinking, and study.

There are two What Are My Rules? games on Pupil Page 5. Both suggest the idea of subtraction as did the exercises at the top of the page. As we shall continue to point out, do not tell pupils how to do these exercises. Let them explore these and similar exercises and determine by the patterns what they are to do. Give them further clues, of course, but try to withhold actual explanations until everything else has failed. Reexamine our suggestions for doing silent arith-

metic on pages 1-5 of the booklet Games and Enrichment Activities.

Drill

Here is a suggestion for a drill sheet:



	long way	short way
3.	5 hundreds, 6 tens, and 3 ones	___
4.	7 hundreds, _ tens, and 9 ones	789
5.	_ hundreds, _ tens, and _ _ ones	456
6.	_ hundreds, _ tens, and _ _ ones	546
7.	3 hundreds, 0 tens, and 7 ones	___
8.	3 hundreds, 7 tens, and 0 ones	___
9.	8 hundreds, 0 tens, and 0 ones	___
10.	_ hundreds, _ tens, and _ _ ones	222

Here are some stories for which pupils can be asked to write one or more headlines:

1. The lighthouse keeper helped 7 sailboats and 6 motorboats in from the sea.
2. The popcorn vendor had 76 bags of popcorn to sell. At the end of the day he had 18 left.
3. Sandy had 6 dimes and 8 pennies. She spent 5 dimes and 5 pennies.

Headline for the Day:

Cindy had 26¢ on Monday. She spent 2¢ each day for bubble gum. Make a chart to show how much money Cindy had left each day until she spent all of her money.

NOTES ON PUPIL PAGE 6

Tasks: Pupils count or use shortcuts to counting to record in the tables the number of blue tiles and green tiles, and the total number of tiles for each of exercises a - y, and A - H.

Purpose: To encourage the use of shortcuts to counting (i.e. multiplication); to give practice in recognizing rectangular representations of numbers; to review addition and subtraction.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Flash cards showing rectangular arrays of square tiles and dots.

Materials for Pupils: Graph paper.

The Lesson:

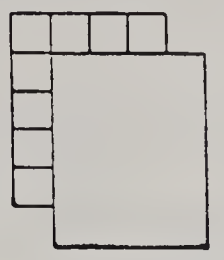
On this page pupils record in the table the number of tiles colored green, those colored blue, and the total number of tiles in each rectangular array.

You will notice that arrays a - y represent the multiplication tables through 5×5.

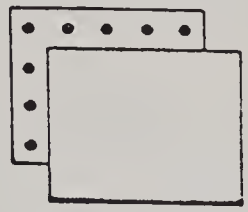
Pupils who develop skill in looking at a rectangular array and quickly responding with the total are well on the way toward mastering the basic facts of multiplication and division.

Many pupils who studied Level B of MATH WORKSHOP are likely to have mastered finding the numbers for the rectangular arrays up to 5-by-5 arrays. To make the tie to muliplication strong, include practice with flash cards for

these 25 facts of multiplication. Mask each card so that only a pair of adjacent edges are exposed:



How many little squares altogether?



How many dots altogether?

Vary the masking procedure — sometimes expose the right and bottom edges, sometimes the left and bottom edges, etc.

Let pupils use graph paper to draw rectangular arrays. Use expressions like "a four-by-five rectangle" and "a one-by-three rectangle." For each drawing, the pupil should write the total number of small squares included in the array. Reverse the exercise by asking pupils to draw rectangular arrays to represent given numbers. Let pupils become aware of the class of numbers which can be represented only by "one-by" arrays. (Except for the number 1, these are the prime numbers. See page 21 of the booklet Games and Enrichment Activities and suggestions below for activities designed to help pupils explore prime numbers.) Help them become aware of which numbers can be represented "◯-by-◯" arrays — these are the square numbers.

Encourage pupils to fill in the Total column in the tables on pupil page 6 before filling in the other columns. Note which pupils recognize the numbers of tiles in arrays a - y and don't need to count. These pupils already have a firm grasp of the idea of multiplication. Those who count one at a time need much help in counting by rows and columns (i.e., by 2's or 3's or 4's, etc.) until they have mastered at least the 5×5 table.

In recording the numbers of green and blue tiles, pupils again need to be reminded of shortcuts.

Help pupils notice the symmetry within the arrays a - y through such questions as:

1. Which array is like b? Answer: (f) c? (k) h? (l) n? (r) etc.
2. What is special about arrays a, g, m, s, and y? (Same number of columns and rows — they are squares.)

Before letting pupils complete work on the arrays labeled A-H, ask them to compare the arrays. Help them notice that the number of blocks in C is the same as the number of blocks in A and B together. This observation may be recorded by writing: $N(C) = N(A) + N(B)$. Similarly:

$N(D) = N(B) + N(C)$

$N(E) = N(\ ) + N(\ )$

$N(F) = N(\ )$

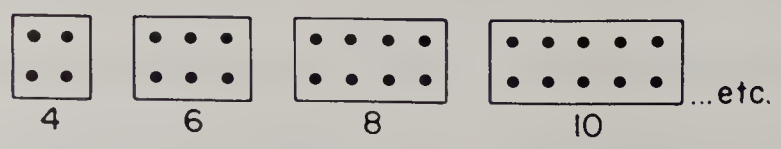
$N(G) = N(\ )$

$N(B) = N(\ ) + N(\ )$

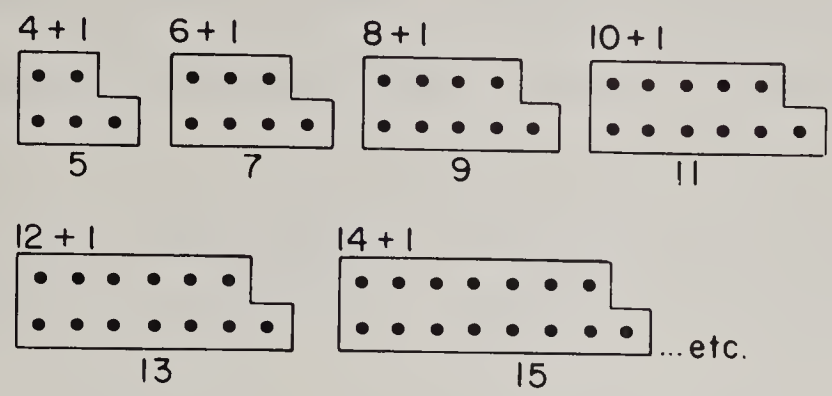
Have pupils complete the above.

Discovering Prime Numbers

Which numbers greater than 3 can be represented by rectangular arrays of dots with more than one dot in each row and each column? Certainly, there are such arrays for all even numbers greater than 3. Let' s draw pictures to show this:

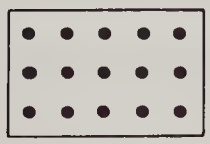


But how about the odd numbers greater than 3? First, let's draw a sequence of pictures that follow a pattern so that we do not skip any odd numbers. We shall copy the pictures of even numbers we have already drawn and then we will add one dot to each:



Which of these displays of dots can be rearranged in rows and columns so that all rows will contain the same number (greater than 1) of dots and all columns will contain the same number (greater than 1) of dots? What about the display of five dots? Seven dots? Try as we will, we cannot find any such arrangement with five or seven dots! Can nine dots be so arranged? (Yes.)

How about 11 dots and 13 dots? (Neither can be so arranged.) How about 15 dots? (Yes.)



Can 17 dots and 19 dots be so arranged? (No.)

Now the point has been reached for individual experiment. Let's make a

list of all the numbers greater than 3 that we have found for which we cannot find a rectangular array of dots with each row and each column containing more than 1 dot:

5 7 11 13 17 19

We call these numbers prime numbers. What is the next larger number that is a prime number? Have pupils find all the numbers less than 50 that are prime numbers.

A prime number is a whole number greater than 1 whose only representation by a rectangular array is a "1 -by-□" or a "□-by-1" array. In other words a prime number is a whole number which has exactly two factors, 1 and itself.

Drill

Consider using as a drill sheet page 5 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for which pupils can be asked to write one or more headlines:

1. Celia had 89 cents. She spent 53 cents.
2. In the first row of Dana' s class there are 4 girls and 3 boys.
3. Karen and Jason counted cars as they walked to school. Karen counted 68 cars and Jason counted 87 cars.

Headline of the Day:

Circle the numbers listed in the total column for exercises a, g, m, s, and y on page 6. What can you say about these numbers?

NOTES ON PUPIL PAGE 7

Tasks: Pupils are to count or use shortcuts to counting to record in the table the number of blue tiles, green tiles, and the total number of tiles for each of the exercises a - r.

Purpose: To review and practice adding and partitioning numbers with some emphasis on multiples of ten; to encourage the use of shortcuts to counting.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Flash cards; Ditto master.

Materials for pupils: Cuisenaire rods; graph paper; Ditto sheets.

The Lesson:

This is a lesson in working with multiples of ten. As recommended for pupil page 6, emphasis on this page should be placed on filling the Total column first. Ask pupils if they can see a simple way to count the total number of tiles in each array. We wish them to look for shortcuts to counting! The chief discovery to be made on this page is that each of the arrays except r is 10 tiles wide. Thus, we can count by 10's as one shortcut. In a through e this is all we need to do to get the total. In j through m we have multiples of 10 and some left over. In n, o, and q, if we can imagine regrouping, we come up with rectangular arrays with 10 in each row.

The counting of the blue and green tiles should be done by looking for familiar arrays. (In exercise r one of the four green regions consists of $(3 \times 3) + 1$ tiles.)

Before the pupils work on this page it may be wise to include further practice

in recognizing arrays using both the flash cards and the activities with graph paper discussed in notes on pupil page 6.

Adding Tens

When pupil page 7 has been completed, have pupils complete the following sentences on a Ditto sheet:

50 + □ = 50	30 + □ + □ = 50	20 + □ + □ + □ = 50
40 + □ = 50	□ + 30 + 10 = 50	50 = 10 + □ + 10 + 10
30 + □ = 50	10 + 10 + □ = 50	50 = □ + □ + 20 + 10
20 + 30 = □	20 + 20 + 10 = □	□ = 10 + 10 + 10 + 20
□ + 40 = 50	□ + 20 + 10 = 50	□ + □ + □ + □ + □ = 50
	□ + □ + 10 = 50	

Remember that in a sentence with frames, you must write the same number in all frames of the same size and shape. For example, the first sentence in the second column must be completed as 30 + 10 + 10 = 50 and not as 30 + 5 + 15 = 50. (This rule of "same number for frames of the same size and shape does NOT mean that frames of different size or shape must be filled with different numbers. See "Enrichment Activity 3" in notes on pupil page 5.)

Cuisenaire Rods

Establish the agreement that the white Cuisenaire rod represents 10. Then have pupils complete the following number sentences and construct models based on this agreement. The diagram shows models for sentences (a) through (f):

(a) 50 - 50 = △

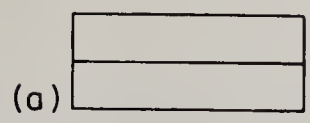
(b) 50 - 40 = △

(c) 50 - 30 = △

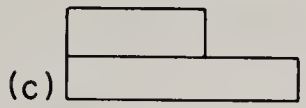
(d) 50 - 20 = △

(e) 50 - △ = 40

(f) 50 - △ = 50

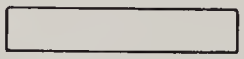
(a) 

(b)

(c) 

(d)

(e)

(f) 

Extend the Patterns

10, 20, 30, 40, , , ,

15, 25, 35, , , , ,

10, 30, 50, , , , ,

0, 20, 40, , , , ,

Numbers Have Many Names

Game 1

Here is a game for the entire class. On the chalkboard, draw the following chart:

Numbers have many names !				
other names for 3 + 2	other names for 10	other names for 11 - 8	other names for 7 + 5 ,	other names for 9 - 9

Use an odd number of columns and select an even number of teams, preferably two. A member of the first team must write in the appropriate column another name for 3 + 2. A member of the next team must write another name for 10, and so on.

The odd number of columns and the even number of teams assure that the team that had 3 + 2, 11 — 8, and 9 — 9 the first time across will have 10 and 7 + 5 the next time and vice versa. Each team scores whenever one of its members gives a correct "other name." Whenever one of its members gives an incorrect other name, the opposing team scores, provided that one of its members spots the error.

Games such as this let each child participate successfully at his own level. Activities based on the idea that numbers have many names (i.e., that there are many ways to "get" a number) help the teacher to estimate the ability of each pupil. They help the teacher to discover who in the class is slow or quick, timid or confident. For example, a front-runner may reveal himself by using $3000 \div 1000$ or $\sqrt{9}$ as other names for 11 — 8. (It is always possible to give the more apt pupils an opportunity to show off their prowess by asking them to write on the board the most difficult looking names for a given number that they can think of.)

Game 2

A variation can be used to make Game 1 more difficult. A member of one team starts a name for the number in question, and the other team must complete it. Another name for 3 —2 might be started as 0 + ___ or as 19 — ___. The next player has to complete the name: 0 + 1, or 0 + 10 — 9, or 0 + 4 —1 —2, etc.

Sometimes, you may wish to begin the other names in order to match the challenge of the situation to the recognized ability of the next player. This is

important to the pupil: to be asked a question that is not too easy nor too hard, a question he can answer if he makes an effort.

Game 3

How many names for each of the numbers from 0 through 10 can you write in the form of ___+___, using whole numbers only?

0	1	2
0 + 0	0 + 1	2 + 0
1 name	1 + 0	1 + 1
	2 names	0 + 2
		3 names

Have pupils record the results in a chart:

Number	Names in form of
0	1
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9
9	10
10	11

This leads to an interesting pattern. It suggests that there will be 21 names for 20 in the form of ___+___. Test the pattern. Does it keep on giving you reliable results?

Game 4

Pupils who finish early the exploration in Game 3 may be asked to explore the same question for names that are in the form ___ + ___ + ___.

0	1	2
0 + 0 + 0	0 + 0 + 1	0 + 0 + 2
(1 name)	0 + 1 + 0	0 + 2 + 0
	1 + 0 + 0	2 + 0 + 0
	(3 names)	1 + 1 + 0
		1 + 0 + 1
		0 + 1 + 1
		(6 names)

3	4	5
0 + 0 + 3	0 + 0 + 4	0 + 0 + 5
0 + 3 + 0	0 + 4 + 0	0 + 5 + 0
3 + 0 + 0	4 + 0 + 0	5 + 0 + 0
0 + 1 + 2	0 + 1 + 3	0 + 1 + 4
1 + 0 + 2	0 + 3 + 1	1 + 0 + 4
0 + 2 + 1	1 + 0 + 3	0 + 4 + 1
1 + 2 + 0	1 + 3 + 0	1 + 4 + 0
2 + 1 + 0	3 + 0 + 1	4 + 1 + 0
2 + 0 + 1	3 + 1 + 0	4 + 0 + 1
1 + 1 + 1	2 + 2 + 0	0 + 2 + 3
(10 names)	2 + 0 + 2	2 + 3 + 0
	0 + 2 + 2	0 + 3 + 2
	1 + 1 + 2	2 + 3 + 0

1 + 2 + 1	3 + 0 + 2
2 + 1 + 1	3 + 2 + 0
(15 names)	3 + 1 + 1
	1 + 3 + 1
	1 + 1 + 3
	2 + 2 + 1
	2 + 1 + 2
	1 + 2 + 2
	(21 names)

The results, once more, are to be recorded in a chart:

Numbers	Names in form of $\frac{\quad + \quad}{\quad + \quad}$
0	1
1	3
2	6
3	10
4	15
5	21
6	?

Multilevel Motivated Practice

Mathematics is full of interesting problems that require extended exercise in recalling the basic combinations and putting them to work. Some of these problems can be stated simply and their solutions require very little maturity. They can be explored by children using the computational abilities they are developing. The solutions are within their reach.

Many of the activities suggested in this guide are truly "multilevel." Some grow out of situations created by manipulative materials used by beginners. Less than obvious questions are posed that can be answered by newcomers to arithmetic.

Most of the suggested activities begin simply. As they are extended, the questions that are asked require more background. If carried as far as possible, many would involve quite sophisticated mathematical ideas. However, the sequence of activities is designed so that a teacher can begin and end with work that is within the capacity of his particular class.

Teachers should consider our suggestions as a source from which to draw problems for discussion and study.

Each activity is presented with emphasis on many ways of providing children with highly motivated drill in the fundamentals of arithmetic. While mathematical ideas that are important in their own right form the bases for these activities, their main purpose is the development of the child' s skill in basic arithmetic for it is only in doing arithmetic correctly that the excitement and fun comes through. We do not wish children to become "mathematical stutters."

Drill

Consider using as a drill sheet page 6 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for which pupils can be asked to write one or more headlines:

1. Cal placed 32 books on the first shelf of his bookcase, 27 on the second shelf, and 33 on the third shelf.
2. 5 tugboats towed 8 big ships into port.
3. Monte picked 7 boxes of berries while Milo picked 5 boxes of berries.

Headline for the Day:

Frances said she could make a square with the pictures for exercises h and i on page 7. How could she do this? What other pictures could you use from exercises a through i to make squares?

NOTES ON PUPIL PAGE 8

Tasks: Pupils count and record in each white box the number of places light would show through.

Purpose: To examine multiplication by using intersecting vertical and horizontal strips.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Demonstration-size, slotted cards of tagboard as described below; overhead projector.

Materials for Pupils: Cuisenaire rods; desk-size slotted cards.

The Lesson:

This and the next several pages review the idea of multiplication (and division) in terms of models which were introduced in LEVELS A and B. We have already encountered (pupil pages 6 and 7) the significance of rectangular arrays as representations of multiplication. This idea will be strengthened in the pictures and experiences that follow. Remember that we wish pupils to become more and more skillful in looking at a rectangular array and giving the number of objects in the array. Before LEVEL C is finished, the average pupil should know the multiplication and division facts through 10×10.

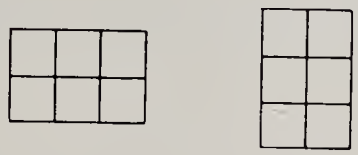
The following models are used in this book as representations of the members of the fact team:

$2 \times 3 = 6$ $\qquad$ $6 \div 3 = 2$

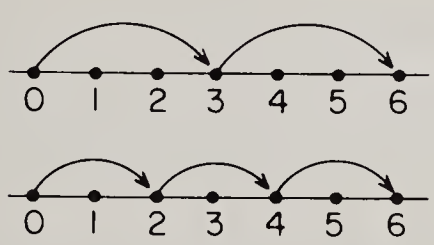
$3 \times 2 = 6$ $\qquad$ $6 \div 2 = 3$

Crossing points of horizontal and vertical lines

Rectangular arrays of counters of dots



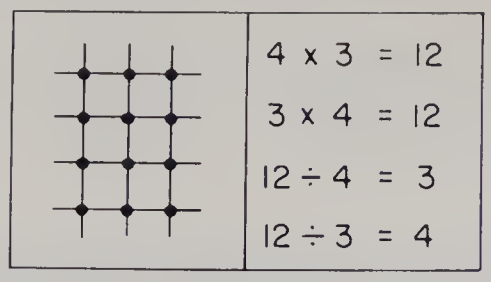
Areas of rectangles

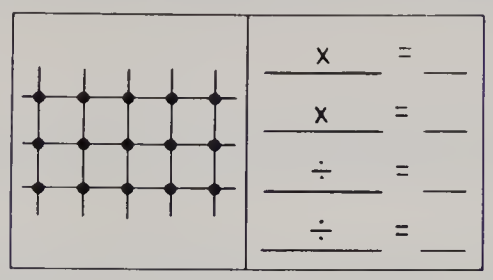


Number line jumps

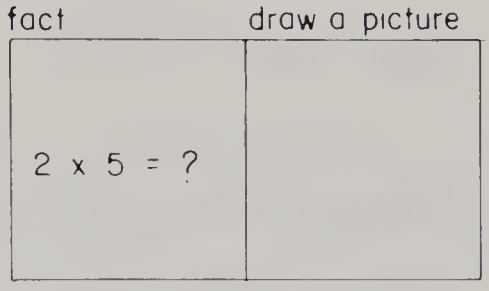
Special emphasis is given on pupil pages 8 through 18 to the crossing points of lines as a model of multiplication.

When pupils have finished these pages, they should be able to interpret such pictures as the following in terms of multiplication-division fact teams:

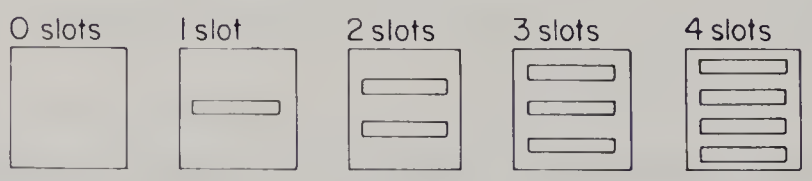




Moreover, if a pupil doesn't remember a particular fact, he should draw a picture to help:



In preparation for pupil page 8 make two sets of slotted cards:



When one card is placed over another so that the slots of one card are horizontal and the other vertical, light will show through the "crossing points." Keep a record on the chalkboard of the number of places where the light shows through.

horizontal slots	vertical slots	crossing points
1	2	2
2	1	2
0	1	0
0	2	0
2	2	4
3	2	
2	3	
3	3	

These actions can be carried out very effectively by holding each card of a pair against a window pane, and then holding the pair against the pane. Or, if an overhead projector is available, place one card on the platform (note the number of streaks of light on the screen) and then superpose the second card. This demonstration will emphasize to the pupil what it is that should be counted in the exercises on pupil page 8.

See pages 62-79 of the booklet Games and Enrichment Activities for multiplication activities involving the use of Cuisenaire rods.

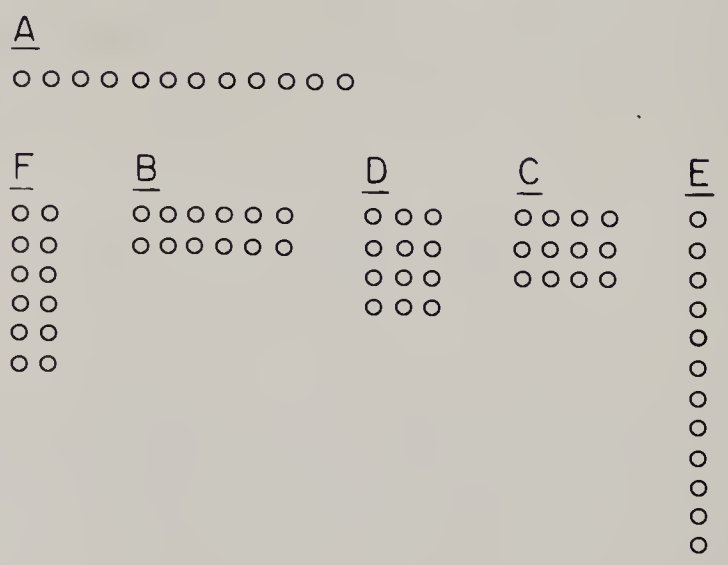
Drill

Here are some stories for which pupils can be asked to write one or more headlines:

1. There were 45 children at the playgound. Twenty-seven of them were boys.
2. Jake's class had a spelling bee. The boys' team made 32 points while the girls' team made 27 points.
3. Sally climbed two ladders. One was 4 feet high. The other was 7 feet high.

Headline for the Day:

Flavio said there were the same number of counters in each picture below. What do you think?



NOTES ON PUPIL PAGE 9

Tasks: In exercises B-D pupils count and record in the white frames the number of vertical or horizonal strips. In the tinted frames pupils record the number of places light would show through. (For exercise A, see the discussion below.) In exercise E and F pupils use the representations provided to record in the tinted boxes either the landing points (exercise E) or the number of triangles (exercise F) specified by the directions.

Purpose: To review and extend understanding of multiplication by exploring several representations of this operation.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Demonstration-size, slotted cards (as described in Notes on Pupil Page 8); number line chunks; tagboard cards picturing objects similar to those pictured in exercise F; overhead projector.

Materials for Pupils: Slotted cards.

The Lesson:

Exercise A provides clues useful in determining what is called for in exercises B-D. In each of the latter exercises we are given four slotted cards. The number in a tinted block is the product of the numbers of slots in the cards joined by the tinted block. The number in a white block (in B-D) is the number of slots in the card touched by the white block.

Each of these exercises (A-D) represents a multiplication Cross Number Puzzle:

A

1	2	2
3	2	6
3	4	

B

2	1	2
4	1	4
8	1	

C

3	2	6
1	5	5
3	10	

D

3	4	12
2	0	0
6	0	

The upper part of this page shows elaborate records of the "crossing points" of slotted cards. As preparation for this page continue the work suggested in Notes on Pupil Page 8, having pupils demonstrate the situations suggested in the chart below. Have pupils anticipate how many crossing points they will find before actually arranging the cards.

Horizontal	vertical	crossing points
1	2	
1	3	
3	2	
2	2	

Record-keeping on pupil page 9 involves (1) recording the number of slots on each card, and (2) indicating for every pair of adjacent cards the number of crossing points that would be observed if the cards were overlapped.

The bottom section of pupil page 9 introduces a technique we shall use often in LEVEL C. We begin with a pictured situation — in the case of this page, jumps of 3 on a number line and two rows of 6 triangles in each row. Pupils are then asked to regard the situation in different ways and to record the number for each way.

Begin this activity with similar examples on the chalkboard until you are satisfied pupils have the idea.

5¢ 1¢	money shown	6¢	twice as much	
	3¢ more		half as much	

apples, 5¢ ea.	2 apples	10¢	2 of each	
oranges, 10¢ ea.	1 of each		2 apples and 3 oranges	

Drill

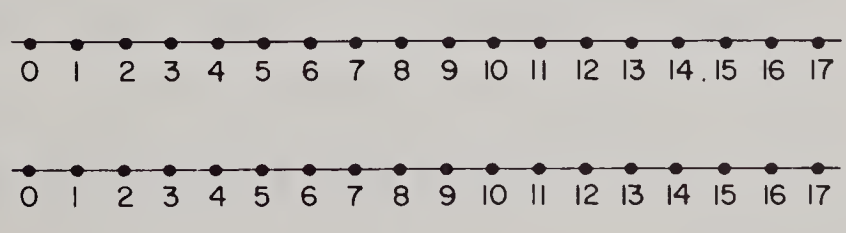
Consider using as a drill sheet page 7 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for which pupils should write one or more headlines:

1. Miss Costanza bought 43 cookies and 28 ice cream bars from the class party.
2. Jason sold limeade and lemonade. He had 7 quarts of limeade and 6 pints of lemonade.
3. A boy planted 8 trees. Three of them died.

Headline for the Day:

John said he could show that two jumps of 3 followed by three jumps of 3 brings you to the same landing point as taking 5 jumps of 3. Use the two number lines below to check to see if John is correct.



NOTES ON PUPIL PAGE 10

Tasks: Pupils use the clues provided in the first four exercises to complete the remaining exercises either by constructing representations or recording in the circles the pertinent information supplied by the given representations.

Purpose: To explore multiplication (and division) using crossing vertical and horizontal lines as representations.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Overhead projector; grease pencil; colored chalk; flash cards (as described below).

Materials for Pupils: None.

The Lesson:

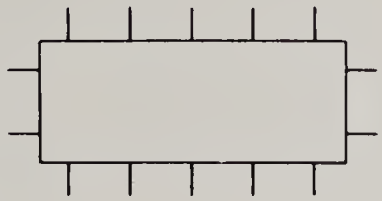
On pupil page 10 (and 11) we have a record of some of Numo's explorations of the number of different ways 1 to 10 lines can be arranged horizontally or vertically, or in some combination of these. Where there is at least one horizontal line and one vertical line, he regarded them as crossing.

One good way to introduce this work is to divide your class into groups of two to four pupils each to work as a team on pupil pages 10 and 11 with no instructions except these: On these two pages are some entries from Numo's math notebook. It appears that some entries are missing. Study what he has done. What are Numo's rules? Can you supply the missing information?

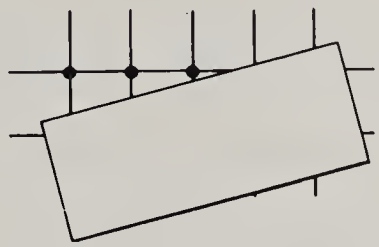
Make sure that one pupil in each group is an able one. This team approach to arithmetic offers pupils a change of pace in their work. It involves cooperation and team work, and at the same time provides the excitement of competition with other teams.

Uncovering Flash Cards

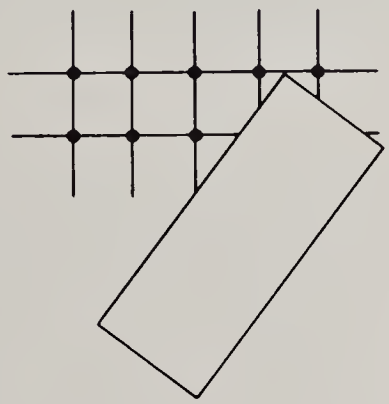
As a follow up to this page, we suggest work with a set of flash cards depicting crossing lines. Cover the crossing points so that only the ends of the horizontal and vertical lines can be seen. Slowly take off the cover card. When can pupils predict the number of crossing points?

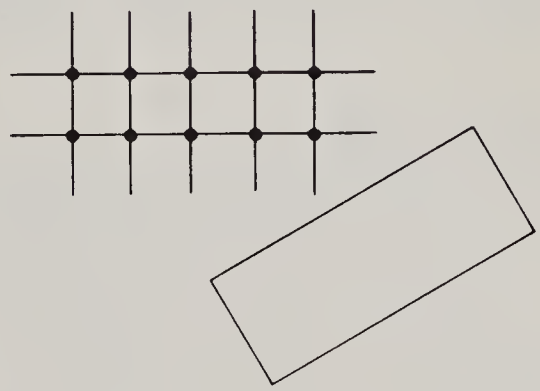


How many lines? _____



How many crossing points?





Drill

Here is a suggestion for a drill sheet:

1.	48 + 10 = ☐	2.	59 − 10 = ☐
3.	70 + 10 = ☐	4.	268 + 10 = ☐
5.	268 + 100 = ☐	6.	268 + 1 = ☐
7.	683 − 100 = ☐	8.	683 − 10 = ☐
9.	683 − 1 = ☐	10.	3 + 10 = ☐
11.	3 + 100 = ☐	12.	3 + 1000 = ☐
13.	728 + ☐ = 748	14.	351 + ☐ = 451
15.	927 − ☐ = 907	16.	732 − 632 = ☐
17.	841 − 831 = ☐	18.	962 − 762 = ☐
19.	573 − 73 = ☐	20.	683 − 603 = ☐

Here are some stories for "headline practice":

1. Jack placed 6 books on each of 4 shelves.
2. The milkman delivered 2 gallons of milk to Mrs. Waters and 4 gallons of milk to Mrs. Sands.
3. Mollie is 6 years old. Manny is 4 years older than Mollie.

Headline for the Day:

Draw 7 lines with 10 crossing points.

NOTES ON PUPIL PAGE 11

Tasks: For the upper part of the page pupils record in the tinted box the number of lines to be used in constructing the required representations. They record in the circles the number of crossing points pictured, and construct the missing representations. For the lower part of the page pupils record in the blue boxes the number of lines used in each representation. In the green boxes they record the number of crossing points for each representation. In the last two exercises pupils must also supply the missing representations.

Purpose: To further explore multiplication (and division) using crossing vertical and horizontal lines as representations.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Overhead projector; grease pencil; colored chalk; flash cards (as described in Notes on Pupil Page 10).

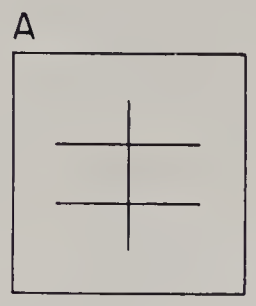
Materials for Pupils: Strips for pupils who are having difficulty in working just with pictures.

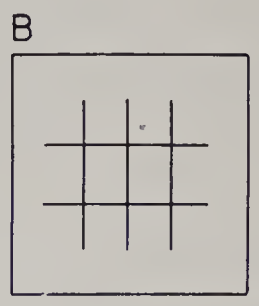
The Lesson:

Draw a table on the board without comment:

	Horizontal lines	Vertical lines	Number of lines	Number of crossing points
A				
B				
C				
D				

Now draw pictures like the following, recording information in the table. (You may wish to use the overhead projector for these drawings.)





	Horizontal lines	Vertical lines	Number of lines	Number of crossing points
A	1	2	3	2
B	2	3	5	6
C				
D				

When you draw the next picture, offer the chalk to the class to see who can fill in the record. Continue with other information and with several pictures.

As a variation, present the numbers in the table and have pupils draw the pictures.

Finally, present problems in which only the number of lines or the number of crossing points, or both, are given. Pupils need to complete the table and draw the appropriate pictures.

	Horizontal lines	Vertical lines	Number of lines	Number of Intersections
E			5	
F				12
G			6	8

Drill

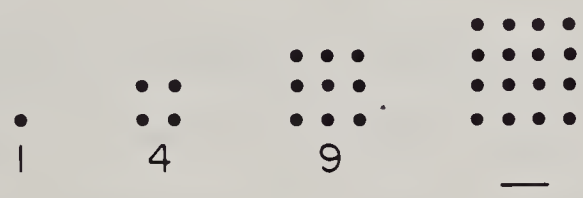
Consider using as a drill sheet page 8 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Toni invited 8 boys and 8 girls to her birthday party. She was able to seat everyone including herself.
2. The milkman delivered 6 gallons of milk and 32 quarts of eggnog to the grocer.
3. Blanche bought 2 sizes of ice cream cones. She also bought 6 different flavors of ice cream.

Headline for the Day:

Extend the pattern two more steps:



NOTES ON PUPIL PAGE 12

Tasks: Pupils record in the tinted regions the missing pairs of letters or the total number of pairs of letters.

Purpose: To examine multiplication in terms of pairing (Cartesian product of sets).

Unifying Ideas: Sets; multiplication and division.

Materials for Teacher: Ditto masters; flash cards (see below).

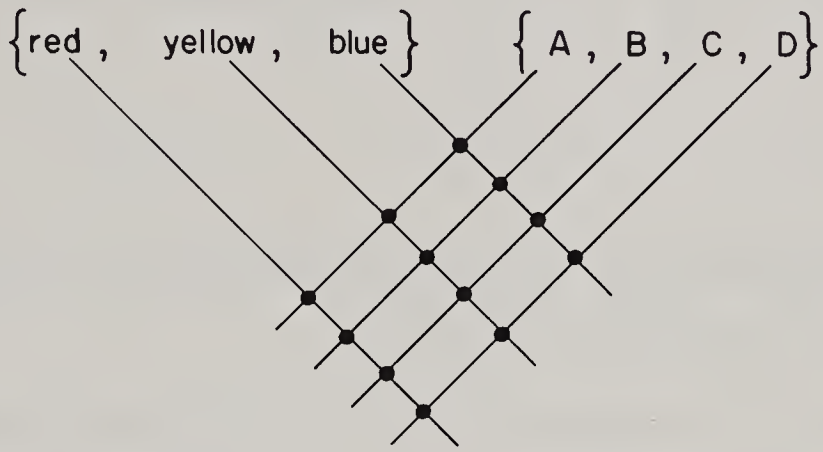
Materials for Pupils: Ditto sheets of simple pairing exercises similar to those on the pupil page and those described below.

The Lesson:

Given a first number and a second number, consider two sets, one of which has the first number for its count and the other of which has the second number for its count. Consider, next, a third set whose members are all the pairs whose first components belong to the first set and whose second components belong to the second set. The third set is called the Cartesian product of the first set by the second set. The number of pairs in the third set is called the product of the first number by the second.

This abstract definition of multiplication is reflected in the illustrations on pupil pages 12 and 13. Let's consider an example. Suppose that we wish to find the product of 3 by 4. According to the definition, we should consider a 3-set and a 4-set, and then consider a third set whose members are all the pairs with first component from the 3-set and second component from the 4-set. For example, let the 3-set be the set consisting of the colors red, yellow, and blue, and let the 4-set be the set consisting of four cardboard squares, one with an A

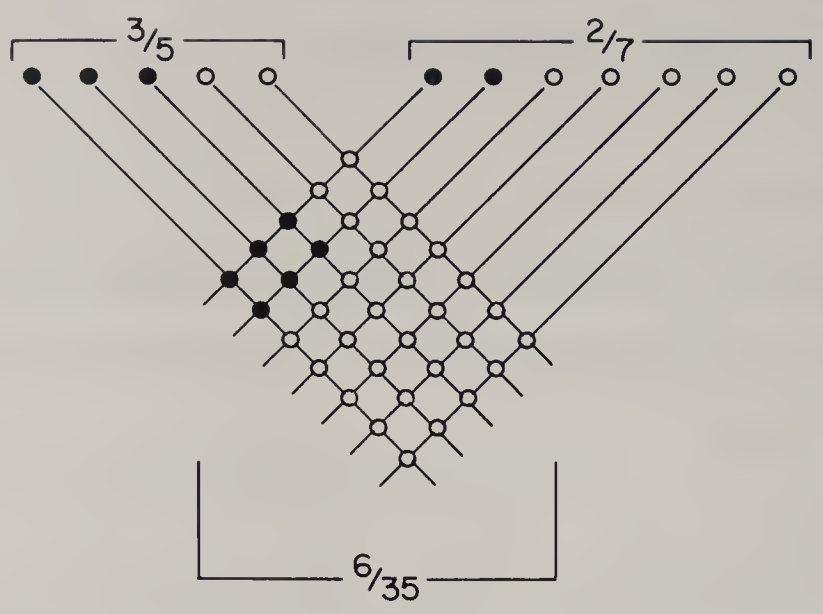
marked on one side, one with a B, one with a C, and one with a D. What are all the possible pairs? Well, you could color the A-square red. Or you could color it yellow. Or you could color it blue. So far, we have three possible pairings of a color and the A-square. As you can see, we also have three possible pairings using the B-square. Likewise, there are three pairings using the C-square and three using the D-square. We can represent all of the numbers of the third set as follows:



Each dot in the array represents one of the pairs in the third set. Since there are 12 such dots, it follows from the definition that 12 is the product of 3 by 4. The same illustration also shows that the product of 4 by 3 is 12.

This definition of multiplication can be used to show that the product of 0 by any number is 0 and that the product of any number by 0 is 0. In either of these cases it is obvious that the number of pairs in the Cartesian product is 0 since pairs cannot be formed if there is no supply of first components or second components.

One of the reasons we have selected this definition for multiplication is that it can be easily extended to multiplication of fractions. For example, here is a way to discover the product of 3/5 by 2/7.



The repeated addition approach to multiplication works well with whole numbers but cannot be reasonably extended to fractions.

It is not our intention that pupils give any verbal description of multiplication in terms of sets. If a pupil wishes to discover the product of 3 by 4, he might first draw 3 parallel lines; then he should draw 4 parallel lines which cross the first 3; then he should count the crossing points. Or, he might draw a 3-by-4 rectangle on graph paper and count the little squares. Or, he might make 4 jumps of 3 on the number line. Etc.

Pairing Skirts and Blouses

You might wish to prepare pupils for the exercises on pupil pages 12 and 13 by having them consider the selection of outfits. These activities are familiar to pupils who have used MATH WORKSHOP, LEVEL B. Make a cardboard Numa for each girl in the class and a Numo for each boy. Give each girl a Ditto sheet with pictures of three blouses and three skirts on it; give each boy a Ditto sheet with pictures of three shirts and three pairs of trousers. Have pupils color each blouse (or shirt) and skirt (or pair of trousers)

a different color, and then have the pupils cut out the costumes.

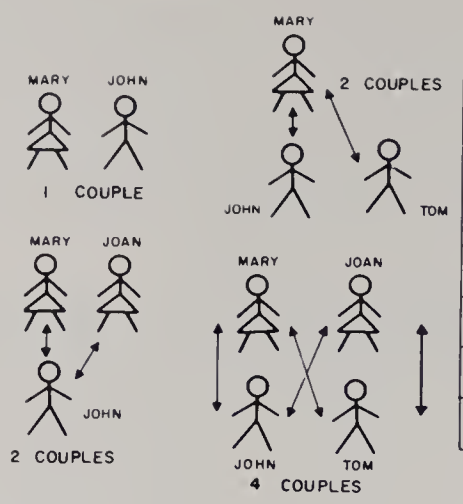
How many complete outfits can you make with one blouse and no skirt? Or with one shirt and no trousers? The answer, of course, is 0. Can you make any complete outfits with one skirt, or one pair of trousers, and no blouse, or shirt?

How many complete outfits can you make with one top and one bottom? (One.) With one top and two bottoms? (Two.) With two tops and one bottom? (Two.) With two tops and two bottoms? (Four.) Let' s make a record to show what we have discovered:

Tops	Bottoms	Outfits
1	0	0
0	1	0
1	1	1
2	1	2
1	2	2
2	2	4
2	3	6
3	2	6

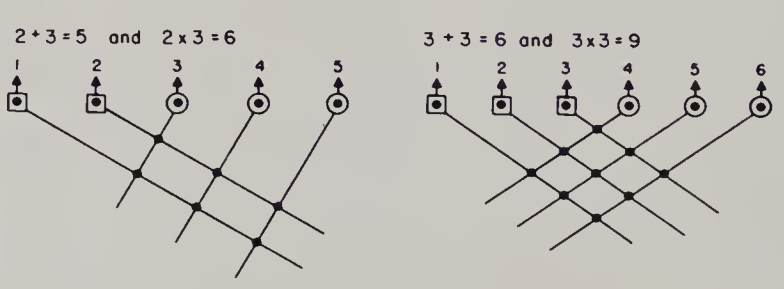
Pairing Boys and Girls

Have Joan come up. Is Joan a couple? (No) Ask John to come up, too. Now, can we make a couple? (Yes) Ask two more girls and two more boys to come forward. There are three girls (Mary, Joan, and Sue) and three boys (John, Tom, and Jim). How many couples can we make choosing from two girls (Mary and Joan) and one boy (John)? How many couples can we make choosing from two girls (Mary and Joan) and two boys (John and Tom)? Draw on the board:



GIRLS	BOYS	COUPLES
1	0	0
0	1	0
1	1	1
2	1	2
1	2	2
2	2	4
3	2	?

As pupils act out situations like these, they get an increasingly clear picture of what we mean when we say that $3 \times 2 = 6$ and that $3 \times 3 = 9$. Children can grasp the difference between multiplication and addition if illustrations are used to show that:



A Shorthand Notation

Develop the agreement that a way to describe what happens in the pairing problems is to write multiplication number sentences:

$$3 \times 2 = 6$$

Read this as "three times two is six" or as "three multiplied by two is six."

Factors-Product

None of the big words or complicated rules that usually accompany the ideas of multiplication or division is mentioned in MATH WORKSHOP. Pupils used to

be told that there was a multiplicand and a multiplier and that these led to a product. They were also told that in division there was a divisor that had to be divided into a dividend in order to get a quotient. No wonder pupils were so often confused. The terminology is enough to discourage most adults.

In MATH WORKSHOP, however, pupils work with the ideas without being burdened with big words. It is possible to refer to these ideas with two words, factor and product. These terms are included in the pupil pages and you should make use of them during class discussions.

A FACTOR
ANOTHER FACTOR
2 × 3 = 6 ← THE PRODUCT

2 × __ = 6 – We have a factar and the praduct.
We can find the missing factar.

2 × 3 = __ – We have twa factars given.
We can find the praduct.

__ × __ = 12 – We have the praduct given.
We can find several pairs af factars that
wauld make this a true statement — (1 and 12,
2 and 6, 3 and 4).

THE PRODUCT
A FACTOR
6 ÷ 2 = 3 ← THE OTHER FACTOR

6 ÷ 2 = __ – We have the praduct and a factar.
We can find the missing factar

6 ÷ __ = 3 – (Same language).

__ ÷ 2 = 3 – We have bath factars given
We can find the missing praduct

Automatic Mastery of Basic Facts

Regular sessions of 3-5 minutes should be spent reviewing the basic multiplication facts learned in connection with the various representations. Such drill will increase the number of automatic responses of pupils. Aim for automatic mastery of the multiplication facts through 10 × 10 by the end of Level C.

Use flash cards like those suggested earlier for addition and subtraction.

(See Notes for Pupil Page 2.) Pupils who miss facts should be asked to rediscover them via representations.

During drill sessions, as the facts are stated, keep a record in a table like the following:

Second factor \ X	0	1	2	3	4	5
5	0	5	10	15	20	25
4	0	4	8	12	16	20
3	0	3	6	9	12	15
2	0	2	4	6	8	10
1	0	1	2	3	4	5
0	0	0	0	0	0	0

First factor

Drill

Here are some stories for headline practice:

1. Linda bought 2 tickets for rides at the carnival. She had 4 different rides to choose from: the roller coaster, the ferris wheel, the merry-go-round, and the airplanes.

   $2 \times 4 = \square$ $\qquad\qquad$ $4 \times 2 = 8$

2. Monica picked flowers for her friend in the hospital. She picked 49 daisies and a dozen tulips.

3. Mrs. Simmons bought 10 gallons of gasoline for her car. She averages 15 miles per gallon of gasoline.

   $10 \times 15 = \square$ $\qquad\qquad$ $15 \times 10 = 150$

Headline for the Day:

Tami bought five pencils for a quarter. One pencil cost 5 cents more than

the others. How much did the more expensive pencil cost?

NOTES ON PUPIL PAGE 13

Tasks: Pupils complete each representation as required and record in the tinted boxes either missing factors or a missing product.

Purpose: To further explore multiplication (and division) in terms of pairing.

Unifying Ideas: Sets; multiplication and division.

Materials for Teacher: Ditto masters.

Materials for Pupils: Cuisenaire rods for pages 62-79 of the booklet Games and Enrichment Activities; Ditto sheets of exercises similar to those on pupil page 13; Ditto sheets of blank multiplication tables.

The Lesson:

Continue with the pairing activities suggested in Notes on Pupil Page 12, recording multiplication number sentences that fit the pictures.

Do examples similar to those on pupil page 13, using the silent arithmetic approach we have referred to earlier (see pages 1-5 of the booklet Games and Enrichment Activities.)

As examples are completed, have pupils keep a record of the two ways of thinking about working with a pair of displays of objects--addition and multiplication:

Table of Sums

Second addend \ +	0	1	2	3	4	5
5						
4						
3		4	5		7	
2						
1				4		
0			2			

First addend

Table of Products

Second factor \ ×	0	1	2	3	4	5
5						
4						
3		3	6		12	
2						
1				3		
0			0			

First factor

Developing such tables simultaneously strengthens the idea that there are different ways to think about obtaining a third number from two numbers.

Simplifying the 100 Facts of Multiplication

Pupils should be given experience in recording the facts of multiplication in a two-dimensional table. Prepare Ditto sheets of blank tables for such studies as indicated below.

The table is designed to include the 100 basic multiplication facts that are essential to progress in arithmetic. Our purpose here is to impress upon children that they really don' t have such a big job if they understand some very important ideas.

Zero as a Factor

If one factor is 0, what do we know about the product? Here are some examples:

$0 \times 3 =$ ___ $\qquad 9 \times 0 =$ ___

$7 \times 0 =$ ___ $\qquad 0 \times 8 =$ ___

$0 \times 0 =$ ___ $\qquad 117 \times 0 =$ ___

Pupils have found that there are 0 pairs if there are 0 first things or 0 second

things to be paired. Thus we now know 19 of the 100 facts of multiplication and can fill in that part of the table easily.

One as a Factor

If one of the factors is 1, what do we know about the product?

$1 \times 8 =$ ___ $\quad\quad$ $0 \times 1 =$ ___

$2 \times 1 =$ ___ $\quad\quad$ $7 \times 1 =$ ___

$1 \times 1 =$ ___ $\quad\quad$ $1 \times 87 =$ ___

If one of the factors is 1 then the other factor and the product are the same number. Again, pictures of pairing have made this idea clear. Now we know another 17 facts of multiplication.

⑨	0	9								
⑧	0	8								
⑦	0	7								
⑥	0	6								
⑤	0	5								
④	0	4								
③	0	3								
②	0	2								
①	0	1	2	3	4	5	6	7	8	9
⓪	0	0	0	0	0	0	0	0	0	0
X	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨

Square Numbers

Ten of the products to be located in the table are often referred to as square numbers. Of course, numbers have no shape at all. But if we think of them in relation to rectangular arrays of dots, we observe that whenever such an array

has the same number of rows as columns and all of the same width, the array has a square shape. The multiplication fact suggested by such an array is one in which both factors are the same number. There are 10 cells in the table that should contain square numbers. Two of these, 0×0 and 1×1, we have already considered.

$0 \times 0 =$ ___	$5 \times 5 =$ ___
$1 \times 1 =$ ___	$6 \times 6 =$ ___
$2 \times 2 =$ ___	$7 \times 7 =$ ___
$3 \times 3 =$ ___	$8 \times 8 =$ ___
$4 \times 4 =$ ___	$9 \times 9 =$ ___

Only 36 Facts to Memorize

Find the appropriate cells in the table for the products in the above examples. They occupy one of the main diagonals of the table. Since we already know that $0 \times 0 = 0$ and $1 \times 1 = 1$, there are only 8 new facts for us to memorize. That leaves us only 56 more facts to consider. One of these is: $3 \times 7 =$ ___, and it has a partner: $7 \times 3 =$ ___. The order in which a pair of numbers is multiplied doesn' t change the product. Each remaining fact has its partner on the other side of the diagonal of square numbers. Make this clear by giving a few examples.

Have pupils fill in the cells as the table is developed. Have them color in lightly all cells that appear <u>above</u> the diagonal of square numbers and all that have 0 or 1 as a factor. The 36 multiplication facts to be memorized are now clearly revealed:

All the 100 basic facts of multiplication and division.

⑨	0	9	18	27	36	45	54	63	72	81
⑧	0	8	16	24	32	40	48	56	64	72
⑦	0	7	14	21	28	35	42	49	56	63
⑥	0	6	12	18	24	30	36	42	48	54
⑤	0	5	10	15	20	25	30	35	40	45
④	0	4	8	12	16	20	24	28	32	36
③	0	3	6	9	12	15	18	21	24	27
②	0	2	4	6	8	10	12	14	16	18
①	0	1	2	3	4	5	6	7	8	9 ← 1x□=□
⓪	0	0	0	0	0	0	0	0	0	0 ← 0x□=0
X	⓪	①	②	③	④	⑤	⑥	⑦	⑧	⑨

The 36 facts in the box and under the steps are the heart of the problems involving memory.

Inside box, above steps — simply change order of factors.
For example, since 6 x 5 = 30, 5 x 6 = 30.

Simplifying the 100 Facts of Addition

Just as we were able to simplify our approach to the 100 multiplication facts by the use of a two-dimensional table and by noting certain properties of multiplication, so are we able to simplify our approach to the 100 addition facts.

Zero as an Addend

When 0 is one of the addends, what do we know about the sum? Examples:

0 + 4 = ___ 0 + 0 = ___

9 + 0 = ___ 2 + 0 = ___

0 + 1 = ___ 0 + 79 = ___

When one addend is 0 then the other addend and the sum are the same number. (Exactly like multiplication by 1.)

One as an Addend

When 1 is one of the addends, what do we know about the sum?

1 + 7 = ___	6 + 1 = ___
9 + 1 = ___	1 + 0 = ___
1 + 1 = ___	2 + 1 = ___

When one addend is 1, the sum is that number which comes after the other addend in the order of counting.

So far we have treated 36 of the 100 facts as instances of general principles. There are 64 more to be considered.

Doubles

Often we use the expression "double it." The results of doubling are numbers that are listed along one of the main diagonals in the table.

0 + 0 = ___	5 + 5 = ___
1 + 1 =	6 + 6 =
2 + 2 = ___	7 + 7 = ___
3 + 3 = ___	8 + 8 = ___
4 + 4 = ___	9 + 9 = ___

Only 36 Facts to Memorize

Of these ten doubling facts, the first two have already been considered. The other eight must be memorized. How many more to go? (64 — 8 = 56. This reminds us of our experience with multiplication.)

Let' s look at the rest of the facts, one at a time. For example, consider: 3 + 7 = ___. Does this have a partner on the other side of the diagonal of doubles? (Yes: 7 + 3 = ___) Does every cell not on this main diagonal have a partner on the other side of the diagonal? (Yes.)

Have pupils fill in all the rest of the cells. Now color in lightly all cells that appear above the diagonal and all that have 0 or 1 as an addend. Again, we have reduced a mass of 100 facts to a manageable group, 36 in all, to be memorized!

The 100 facts of addition and subtraction.

+	0	1	2	3	4	5	6	7	8	9
9	9	10	11	12	13	14	15	16	17	18
8	8	9	10	11	12	13	14	15	16	17
7	7	8	9	10	11	12	13	14	15	16
6	6	7	8	9	10	11	12	13	14	15
5	5	6	7	8	9	10	11	12	13	14
4	4	5	6	7	8	9	10	11	12	13
3	3	4	5	6	7	8	9	10	11	12
2	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9

The 36 facts that must be committed to memory.

next number (counting)

□+0=□

Try to direct the activity so that the parallelisms that exist between the multiplication and addition tables are obvious. The important fundamental similarities are:

I. Multiplying a number by 1 or adding 0 to it give the number itself as product or sum.

$1 \times 5 = 5$ $\qquad$ $5 + 0 = 5$

$6 \times 1 = 6$ $\qquad$ $0 + 6 = 6$

$1 \times 7 = 7$ $\qquad$ $7 + 0 = 7$

Later, in algebra, pupils will learn a language that will help them express these two ideas in the following way:

For each $\underline{x}$, $\underline{x} \times 1 = 1 \times \underline{x} = \underline{x}$

and $\underline{x} + 0 = 0 + \underline{x} = \underline{x}$.

II. The order in which a pair of factors or addends is considered does not effect the outcome.

$2 \times 3 = 3 \times 2 = 6$ $\qquad$ $2 + 3 = 3 + 2 = 5$

$6 \times 7 = 7 \times 6 = 42$ $\qquad$ $6 + 7 = 7 + 6 = 13$

$9 \times 4 = 4 \times 9 = 36$ $\qquad$ $4 + 9 = 9 + 4 = 13$

In algebra pupils will learn to express these ideas in the following shorthand:

For each $\underline{x}$, for each $\underline{y}$, $\underline{x} \times \underline{y} = \underline{y} \times \underline{x}$

and $x + y = \underline{y} + \underline{x}$.

Drill

Consider using as a drill sheet page 11 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Mr. Jointly needs 1 gallon of oil for the engine of his car. The service station attendant brought the oil in quart cans.

   $\square$ quarts = 1 gallon $\qquad 1 \times 4 = 4 \qquad \square \times 4 = \bigcirc$

2. Fernando took a speed test in handwriting. He wrote 49 letters in the first minute. He wrote 57 letters in the second minute.

3. Lydia planted some small pine trees. She planted 3 rows of trees. She put 8 trees in each row.

Headline for the Day:

Sal paid 3¢ for each of 4 pencils. He gave the clerk 2 dimes and 1 nickel.

NOTES ON PUPIL PAGE 14

Tasks: In each of exercises A-D pupils use the representations to record in the tinted boxes either the missing factor or the missing product. In exercise E pupils use the clues provided (ghosted numbers and arrows) to record the required products in the tinted boxes.

Purpose: To further extend understanding of multiplication (and division) using crossing vertical and horizontal lines as representations.

Unifying Ideas: Multiplication and division.

Materials for Teacher: Overhead projector; transparencies as described below.

Materials for Pupils: None.

The Lesson:

Throughout this book there are many opportunities provided to reexamine ideas that have been studied earlier. Each time the setting is changed, however, so that pupils really need to think about the new situation. Memory alone will not suffice. Another important effect of such diversity of activity is that interest is kept high as pupils look for clues in trying to understand the new situations. Shouts of "Ohhh, I get it." and similar "Ah hah!" expressions of discovery occur frequently in such a stimulating environment.

Pupil page 14 suggests a visual aid if you wish to make one. The mathematical sentences in A-E are records of stacking clear plastic or glass sheets marked with horizontal and vertical lines and determining the number of crossing points. Pupils are asked in C-E to imagine these actions. In E they are asked to record the numbers of crossing points if each two adjacent sheets were placed one on the other. You can prepare them to use their imagination by presenting some of the

exercises on an overhead projector. Draw parallel lines on acetate sheets and use these sheets as transparencies.

You may wish to ask pupils to write all of the multiplication sentences represented in E on another sheet of paper.

Practice on Multiplication Facts

Give practice in multiplication in tables like the following:

×	0	2	4
4			
2			
0			

×	1	3	5
5			
3			
1			

×	0	2	4
5			
3			
1			

Notice that in these examples we are also exploring the questions:

1. Given a pair of even factors, what can you say about the product?
2. Given a pair of odd factors, what can you say about the product?
3. Given one even and one odd factor, what can you say about the product?

Drill

Here is a suggestion for a drill sheet:

Fill in the unshaded squares in the addition table.

+	0	1	2	3	4	5	6	7
0								
5								
10								
15								
20								
25								
30								
35								

Here are some stories for headline practice:

1. Mrs. Toll changed the oil in her 4 trucks. Each truck needs 8 quarts of motor oil.

$$8 + 8 + 8 + 8 = \square \qquad 4 \times 8 = 32$$

$$8 \times 4 = \square$$

2. Loni bought eight 5-cent stamps.

3. Butch put 25 candles on his dad' s birthday cake. His dad is 49 years old.

Headline for the Day:

Circle the number which doesn' t belong in the pattern.

1, 2, 4, 8, 16, 32, 62

NOTES ON PUPIL PAGE 15

Tasks: Pupils complete the table for each given condition by recording the missing addends.

Purpose: To review subtraction in terms of imagining the partitioning of displays of objects.

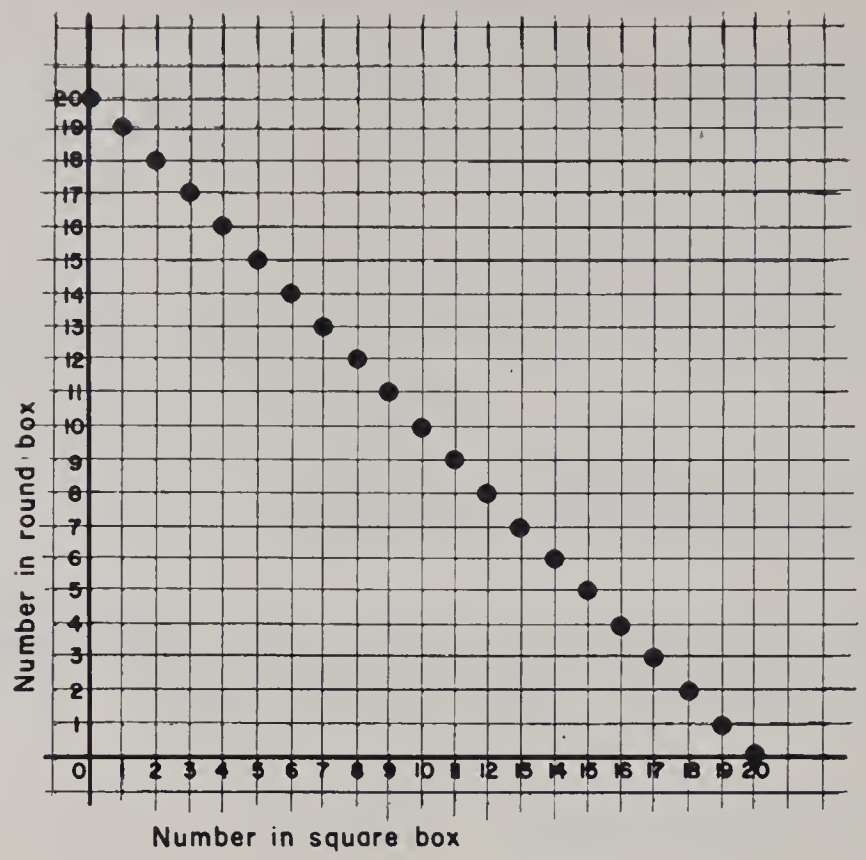
Unifying Ideas: Structure; addition and subtraction.

Materials for Teacher: Two demonstration-size boxes (square and circular); marbles or counters; number line chunks; graph board.

Materials for Pupils: Cuisenaire rods or counters.

The Lesson:

The exercises on this page provide excellent practice in subtraction and many opportunities to gain insights into subtraction. As preparation for the first exercise (If Bill had 20 marbles...), you may wish to draw a graph on a graph board. Start by drawing a horizontal number line and a vertical number line as shown below. Then suppose that Bill put 15 marbles in the square box and 5 marbles in the round box. Put one finger of your right hand on the 15 on the horizontal number line and one finger of your left hand on the 5 on the vertical number line. Move the left hand horizontally to the right and the right hand vertically upward. The paths will cross at a point. Put a large dot at that point. Let pupils "plot" other points, each point representing one of the partitions of 20 objects. Here is what the graph looks like when all 21 points have been plotted:



The graph provides interesting insight into addition (and subtraction). For example, the fact that the graph slopes downward from left to right shows that, given a pair of variable quantities whose sum is constant, if one quantity increases then the other must decrease, and conversely. Also, if one increases by a certain amount, the other must decrease by that same amount, and conversely.

This last idea is useful in doing the exercises on pupil page 15. For example, consider the second table. Suppose that the pupil has written a 40 for his first answer. He then notes that the second problem tells him that there are 20 marbles in the square box, an increase of 10 over the previous case. Hence, there should be a decrease of 10 marbles in the round box. To find the second answer, the pupil merely subtracts 10 from 40.

When pupils have completed pupil page 15, they should be ready for questions such as the following:

Suppose that Bill had 20 (50, 75, 102) marbles and wished to put them all in

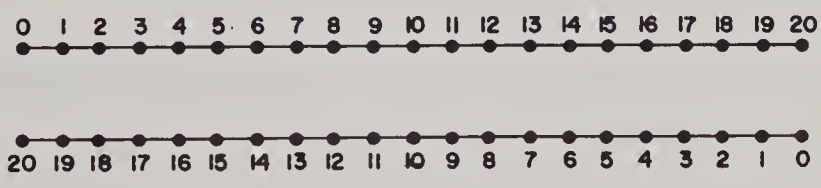
the two boxes so that the square box would contain 2 more than the round box. How many should he put in each? (Answers: (11, 9); (26, 24); can' t be done; (52, 50).

The first of these questions (the case of 20 marbles) can be solved mechanically by using the graph discussed above. Plot some of the points which represent situations in which the square box has 2 more marbles than the round box--(2, 0), (3, 1), (4, 2), (5, 3), etc. The line of plotted points will head toward the point (11, 9), a point which belongs to the previous graph. This common point represents two situations--a total of 20 marbles for both boxes, and 2 more marbles in the square box.

Can a pupil predict that in the case of 75 marbles, it is impossible to have 2 more in the square box? Suppose that there were an even number of marbles in the square box. Then there would be an even number in the round box. Since the sum of two even numbers is even and since 75 is not even, there can' t be an even number of marbles in the square box. Suppose that there were an odd number of marbles in the square box. Then...

Number Line Chunks

The problems on pupil page 15 can also be solved by using pairs of number line chunks properly aligned. For example, in the case of the first table, all we need is a pair of chunks such as the following:



Each pair of dots in vertical alignment represents a partition of the 20 marbles.

Drill

Consider using as a drill sheet page 12 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. There are 29 children in a third grade class at Zabrachburg Elementary School. 14 of the children are girls.
2. Anabru Jr. High School' s football team scored 6 touchdowns. Each touchdown is worth 6 points.
3. When mother weighed herself while holding the baby, she weighed 151 pounds. When she put the baby down and weighed herself, she weighed 119 pounds.

Headline for the Day:

What are some of the points that you could land on if you start at 2 on the number line and make jumps of 3 to the right?

NOTES ON PUPIL PAGE 16

Tasks: In each of exercises A-J pupils follow the directions at the top of the page and record in the tinted boxes all possible 2-digit numbers which can be named with the digits in the given lists. Pupils also record in the blanks the number of 2-digit numbers they have named. In exercise J pupils record in the blanks all possible blouse-skirt pairs and the number of such combinations.

Purpose: To explore multiplication via pairing.

Unifying Ideas: Numeration; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: Cuisenaire rods.

The Lesson:

(See Notes for Pupil Page 17.)

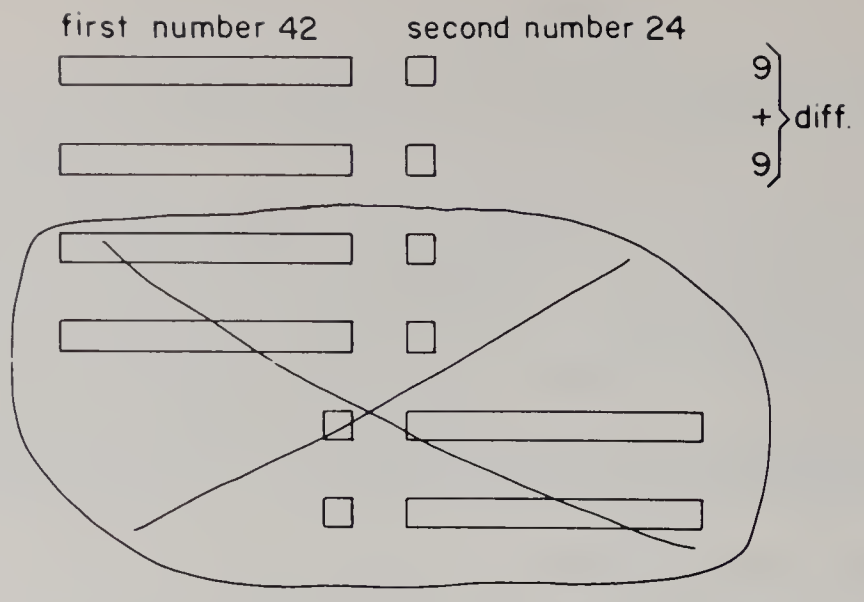
When pupils have completed this page, you may wish to ask questions such as the following:

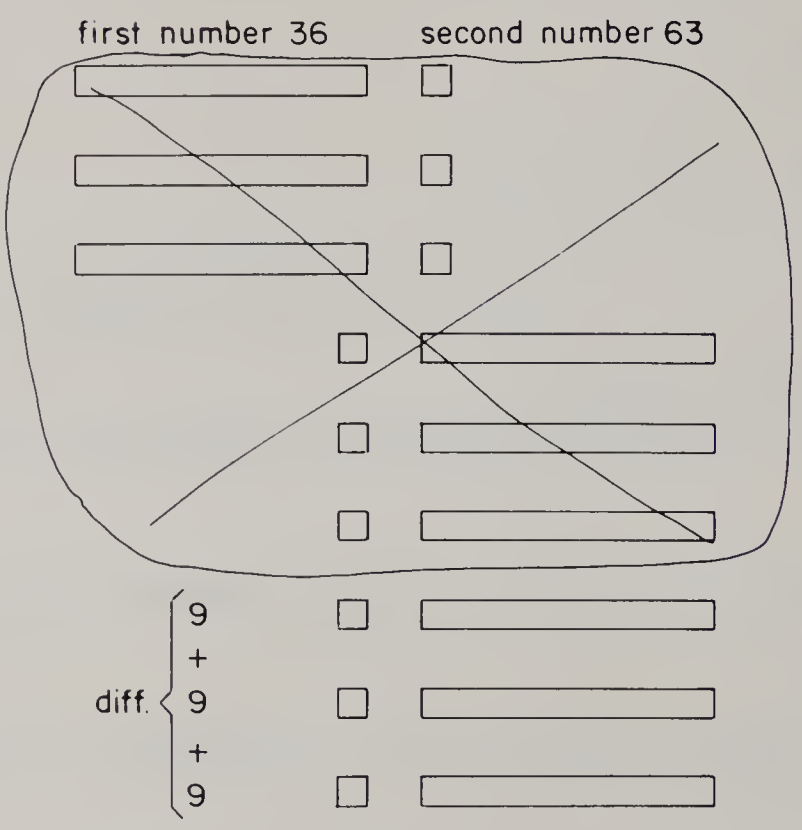
(1) How many 2-digit numbers are there altogether? (Answer: 90, because $9 \times 10 = 90$)

(2) How many 3-digit numbers are there altogether? (Answer: 900, because $90 \times 10 = 900$)

(3) Write a 2-digit number and then write the 2-digit number obtained by reversing the digits of the first one. (Exercises G and H should help prepare for this one.) Subtract the smaller from the larger. Is the result a number in the multiply-by-9 table? (Answer: It should be.)

You can get insight into (3) by using Cuisenaire rods to represent the numbers. Here are two examples:





Also, question (3) suggests a fourth question relating to the result of adding the two numbers. We leave this for you to explore.

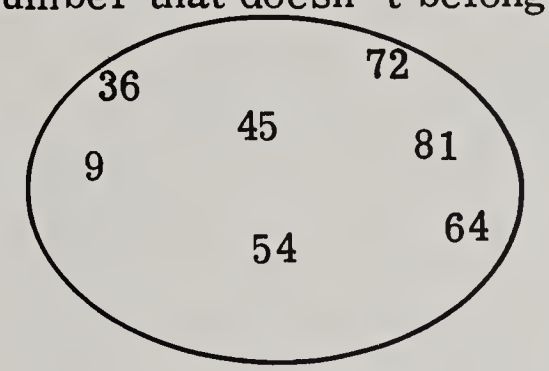
Drill

Here are some stories for headline practice:

1. Joan and Norman read 40 books last week. Joan read 18 of the books.
2. Rosie bought a box of chocolates. There were 15 chocolates arranged 3 in each row.
3. Alma took a jet airplane to San Francisco. She counted 34 passengers entering at the front of the jet and 81 passengers entering at the rear of the jet.

Headline for the Day:

Circle the number that doesn' t belong with the others:



NOTES ON PUPIL PAGE 17

Tasks: In exercises I-IX pupils follow the directions at the top of the page and record in the tinted boxes all 2-letter words which can be constructed with the letters in the given lists. Pupils also record in the blanks the numbers of 2-letter words they have constructed. In exercise X pupils record in the blanks all possible food combinations described in the "advertisement" and the total number of such combinations. (Since the sandwiches are free, it is reasonable to agree that one should not be piggish and use more than one condiment on a sandwich.)

Purpose: To continue the exploration of multiplication via pairing.

Unifying Ideas: Multiplication and division.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

Pupil pages 16 and 17 provide four more experiences with the idea of multiplication. We are again involved with finding all pairings of the members of two sets. (Compare exercise J on pupil page 16 with exercise X on pupil page 17.)

It would be well to divide the class into teams of 3 or 4 pupils to work on these pages on their own while you supervise their work. During this time you may prefer to work with individuals or small groups needing closer supervision or more direct instruction on work studied up to this point. While we encourage the discovery approach in MATH WORKSHOP, it is evident that there will be times when some pupils need more direct explanations. Do not hesitate to give such help when you feel a need for it.

Practice With Addends and Sums; Factors and Products

Each addition or multiplication fact involves a triple of numbers--(addend, addend, sum) or (factor, factor, product). With just two exceptions, if you are given any two components of the triple involved in a fact, you can find the third component. For example, if you are given the addition triple (2, ___, 8), you know that the missing component is 6 because 2 + 6 = 8 and there is only one number which can be added to 2 to give the sum 8. However, if you are given either of the multiplication triples (0, ___, 0) or (___, 0, 0), you cannot determine the missing component because the product of <u>any</u> number with 0 is 0.

Put several tables on the chalkboard or on Ditto sheets for pupils to complete:

	Addend	Sum	Addend
A		8	7
B		6	3
C	2	9	
D	5		6
E	5	15	

	Sum	Addend	Addend
I	3		2
II		0	8
III	20	18	
IV		5	10
V	5	0	

	Factor	Product	Factor
a	3		4
b	2	6	
c		10	5
d	8		2
e		1	

(On rare occasions, we need only be given one component of the triple to determine the other two. This is the case in the last example above. Provided that we are limited to whole numbers, we can complete the last line above, because (1, 1, 1) is the only triple of whole numbers in which the product is 1. The same is true of addition if we know that the sum is 0, since (0, 0, 0) is the only addition triple of whole numbers in which the sum is 0.)

	Product	Factor	Factor
(1)	18		
(2)		4	5
(3)	36	6	
(4)		1	2
(5)	7	1	

Here are some examples in the mathematical shorthand of addition and multiplication. In each case, one component of the triple is missing: an addend, a sum, a factor, or a product. Which is it? What number would convert the sentence in a true one?

$3 + 4 = ___$ $\qquad$ $2 \times 5 = ___$

$7 + ___ = 9$ $\qquad$ $4 \times ___ = 8$

$___ + 8 = 12$ $\qquad$ $___ \times 3 = 9$

$9 = 4 + ___$ $\qquad$ $___ = 6 \times 2$

$12 = ___ \times 3$ $\qquad$ $___ = 0 + 7$

$___ = 15 \times 1$ $\qquad$ $20 = ___ \times 10$

$7 = 1 + ___$ $\qquad$ $2 \times ___ = 18$

$3 \times 5 = ___$ $\qquad$ $12 = ___ + 7$

$___ = 4 \times 4$ $\qquad$ $10 + ___ = 15$

Drill

Consider using as a drill sheet or as test on basic addition facts page 14 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Mr. Hanson' s third graders cut out 58 pictures of storybook people and places. They colored 37 of them with crayons and painted the rest.
2. In a ring toss game Joshua had the best score. He made 4 on his first turn, 4 on his second turn, and 3 on his third turn.
3. Rudy placed 24 books on 3 shelves of his bookcase so that each shelf contained the same number of books.

Headline for the Day:

Jacques decided to have some dessert. He could select any one of 4 flavors of ice cream--licorice, asparagus, tomato, or cabbage. He also had his choice of any one of 3 syrups--watermelon, lettuce, or cauliflower. Make a list of the combinations he could choose from.

NOTES ON PUPIL PAGE 18

Tasks: Pupils use the information provided by each line of the table to record in the white blank on that line the letter of the array whose description is given.

Purpose: To practice partitioning and multiplying, and factoring and adding.

Unifying Ideas: Addition and subtraction; multiplication and division.

Materials for Teacher: Flash cards; Ditto masters.

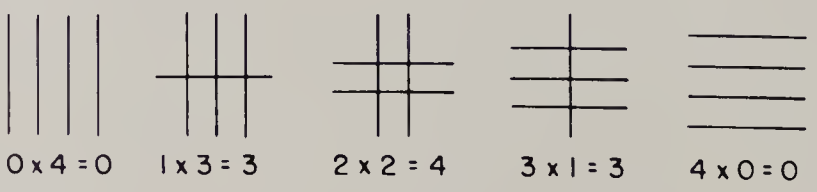
Materials for Pupils: Ditto sheets.

The Lesson:

In preparing pupils to get the most from this page give them practice exercise such as the following (perhaps on Ditto sheets):

Using a total of 4 lines, draw as many different multiplication pictures as you can. State the multiplication fact for each picture.

Answer:



Or, give them oral exercises such as the following:

I have used a total of 8 lines in drawing a multiplication picture. What are the possible number of crossing points? (Answer: 0, 7, 12, 15, 16). I drew a multiplication picture with 20 crossing points. How many lines might I have used? (Answer: 21, 12, 9)

A pupil might solve the problems on pupil page 18 by writing next to each sketch the number of horizontal lines and vertical lines and the number of crossing points. Then, to match the sketch with the tabular data, all he needs to do is look for the sketch' s pair in the table. Of course, this is an acceptable procedure but it doesn' t afford the pupil as much opportunity in creative thinking as he would have if he worked from the table to the picture. For example, in finding a sketch for the pair (10, 24), he can run through in his mind all the 2-component partitions of 10 (10 + 0, 9 + 1, 8 + 2, ...) in order to find one the product of whose components is 24 ($10 \times 0 = 0$, $9 \times 1 = 9$, $8 \times 2 = 16$, $7 \times 3 = 21$, $6 \times 4 = 24$), and then look for a 6-by-4 sketch. Or, he might run through the 2-component factorizations of 24 (1×24, 2×12, 3×8, 4×6,) in order to find one the sum of whose components is 10 ($1 + 24 = 25$, $2 + 12 = 14$, $3 + 8 = 11$, $4 + 6 = 10$), and then look for a 4-by-6 sketch. Or, he might run through the partitions and the factorizations simultaneously.

Note that the prime numbers in the exercises (7, 19, and 11) can be represented only by " 1-by-___" sketches (C, V, and Q). Ask pupils if they can suggest other numbers which can be represented only by " 1-by-___" sketches.

Test on Multiplication Facts

You may wish to prepare several Ditto sheets to test pupil mastery of the multiplication facts. Any of the following approaches might be used:

1. Fill in a multiplication table.

X	0	1	2	3	4	5	6	7	8	9
9										
8										
7										
6										
5										
4										
3										
2										
1										
0										

2. Fill in a multiplication table in which the order of factors is mixed or in which the factors are listed in decreasing order.

X	0	2	4	6	8	1	3	5	7	9
4										
5										
6										
3										
7										
0										
1										
8										
2										
9										

X	9	8	7	6	5	4	3	2	1	0
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

3. Test the facts presented in this form:

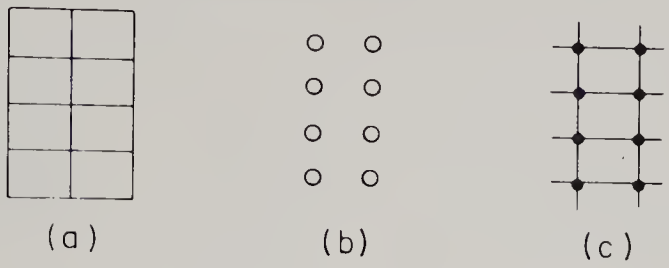
a. $2 \times 4 = ___$

b. $4 \times ___ = 8$

c. $___ \times 8 = 0$

etc.

In all such tests pupils should have scratch paper or use the back of the sheet to draw pictures when they are unsure or make an error which you detect. Remind them of these.



Flash Cards for the Multiplication Facts

Suggestions given earlier (see Notes on Pupil Page 2) for mastering the addition and subtraction facts with the help of flash cards may be applied now to multiplication and to division.

Various Notations for Division

Here are the three common notations for division:

Example: 12 divided by 3 = ?

$3\overline{)\,12}^{\,?}$ $\qquad 12 \div 3 = ? \qquad \frac{12}{3} = ?$

We have, of course, already anticipated division in our experience with multiplication. Actually, a division problem involves a multiplication triple for which a factor component is missing.

Each of the sentences: $3 \times ___ = 12$ and: $___ \times 3 = 12$ asks the same question as is posed in each of the various notations given above.

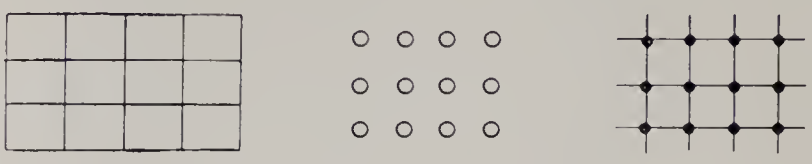
It is important in considering the question asked in each of these notations to realize that in each case we are looking for a missing factor, and to recognize that any picture which represents a multiplication fact also represents the other

three members of the fact team. Thus, each of the sentences:

$3 \times 4 = 12$ $\qquad$ $12 \div 4 = 3$

$4 \times 3 = 12$ $\qquad$ $12 \div 3 = 4$

is a proper headline for any of the following pictures:



$3 \times 4 = 12$... If we have a rectangular array of 3 rows and 4 columns, we have 12 blocks (or dots or crossing points).

$4 \times 3 = 12$... If we have a rectangular array of 4 columns and 3 rows, we have 12 blocks (or dots or crossing points).

$12 \div 4 = 3$... If we have a rectangular array of 12 blocks and there are 4 columns, we have 3 rows.

$12 \div 3 = 4$... If we have a rectangular array of 12 blocks and there are 3 rows, we have 4 columns.

Drill

Here is a suggestion for a drill sheet:

Rules:

(a) Sub 2 $\qquad$ 21, 19, 17, , , , , , ,

(b) Sub 5 $\qquad$ 48, , , , , , , , ,

(c) Sub 10 $\qquad$ 83, , , , , , , , ,

(d) Sub 3 $\qquad$ 37, , , , , , , , ,

(e) Sub 4 $\qquad$ 60, , , , , , , , ,

(f) Sub 6 $\qquad$ 71, , , , , , , , ,

(g) Sub 5 126, , , , , , , ,

(h) Sub 7 100, , , , , , , ,

(i) Sub 6 100, , , , , , , ,

(j) Sub 8 100, , , , , , , ,

(k) Sub 7 145, , , , , , , ,

(l) Sub 9 100, , , , , , , ,

(m) Sub 8 539, , , , , , , ,

Here are some stories for headline practice:

1. Eighteen children went to a party. In one of the games there were 9 on each team.
2. Bill gave 12 cookies to his 6 friends. Each friend received the same number of cookies.
3. Thirty-seven people were awaiting the arrival of the express train. Sixteen of these people were children.

Headline for the Day:

Diane and her father planted 4 rows of tulip bulbs. In each row they planted 5 tulip bulbs. Some of the tulips turned out to be red and the rest of them were yellow. Make a table to show how many could have been red and how many yellow.

NOTES ON PUPIL PAGE 19

Tasks: Pupils record in the tinted boxes and blanks the various partitions implied by the stories and the accompanying suggestions.

Purpose: To use addition and subtraction in several applications.

Unifying Ideas: Addition and subtraction; measurement; functions and relations.

Materials for Teacher: Colored chalk; graph board; coins (see below).

Materials for Pupils: Graph paper.

The Lesson:

What can you say? problems present situations in which the answer is not obvious. Conditions are given that render it possible to make a number of true statements. If the situation is explored and a record kept, it is possible to make rather conclusive statements, but not until a pattern is discovered.

As preparation for this page, put some coins in your hand and close your fingers over them. Say: In my hand I have 9¢ in coins. Can you be sure what coins I am holding? (No; you could have 9 pennies or you could have a nickel and 4 pennies.) Can you say anything for sure about the coins I am holding?

If pupils have not played this kind of game before, you might ask some of the following questions:

Can you say for sure whether or not I have any dimes? (Yes.)

Can you say for sure that I have a least 5 coins? (Yes.)

Can you say for sure that I do not have more than 9 coins? (Yes.)

Can you say for sure that I have at least one nickel? (No.)

Can you say for sure that I have at least 4 pennies? (Yes.)

(In this game of What can you say? attention to precise expression is very important.)

Can you say for sure that I have either 5 coins or 9 coins? (Yes.)

Change the coins in your hand: Now, the coins I am holding are worth 13¢ altogether. What can you say for sure about these coins? Some acceptable answers are:

You have at least 3 pennies.

You have at least 4 coins.

You do not have more than 13 coins.

If you have 5 coins, they are 2 nickels and 3 pennies.

If you have more than 9 coins then you have 13 pennies.

If you have 4 pennies then you have at least 8 pennies.

Without telling the children what coins you actually have in your hand, change the question: Tell me some combinations of coins I could have.

After a few have been suggested, ask: How many different combinations of coins are there that are worth 13¢? Make a chart:

Dimes	Nickels	Pennies	Total No. of Coins
		13	13
	1	8	9
	2	3	5
1		3	4

There are 4 possible combinations.

Similar problems that you may wish to explore with the class as preparation for pupil page 19 include:

1. Tom is twice as old as his brother Bob. Bob is either 7, 8, or 9 years old.

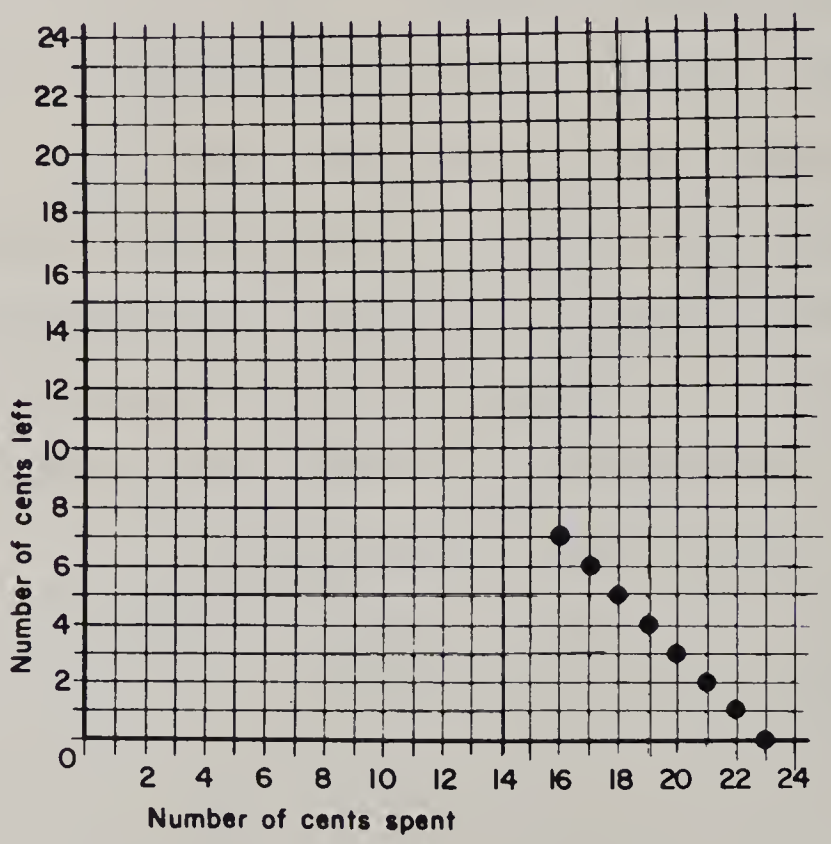
2. I have 15¢ made up of an odd number of coins. What coins do I have?

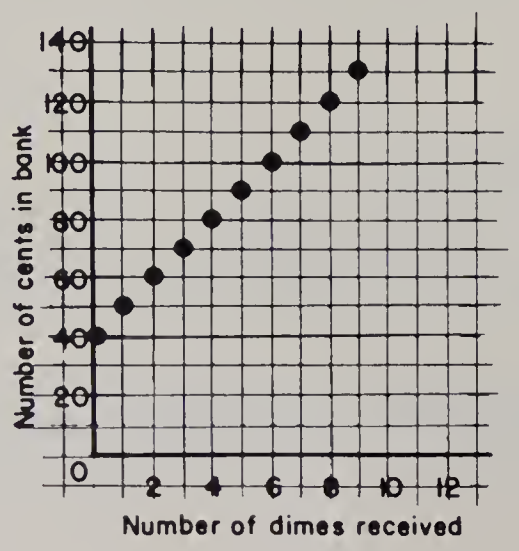
3. I gave the clerk a quarter and he gave me three coins in change. What can you say?

Graphs

What can you say? problems lend themselves easily to graphing. (See Notes for Pupil Page 15.)

Here are graphs for the problems on pupil page 19:





Make these graphs on a graph board and let pupils make them on graph paper. They should understand that each point of the graph represents a possible situation which meets the special conditions of the problem. To test this understanding, put your finger on various points and ask what the points tell. (If he received 6 dimes for his birthday then he had 100 cents in his bank.)

Drill

Consider using as a drill sheet or as test on subtraction facts page 15 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Caleb paid 30 cents for 5 apples.
2. Seven boys and eight girls came to the party at Polly' s house.
3. Seventeen boys from East School are in Little League. Twenty-four boys from the South Elementary School are also members of Little League.

Headline for the Day:

Duane had a quarter. He spent less than 13 cents. What can you say about this situation? Make a table.

NOTES ON PUPIL PAGE 20

Tasks: In exercises A-L pupils record in the tinted areas number sentences which are suggested by each of the representations. In the game in exercise M pupils use the clues provided in the first three columns or the pattern indicators at the extreme right of the table to continue the pattern by making appropriate entries in the remainder of the table.

Purpose: To perfect skills in interpreting representations of addition and in recording observations; to prepare for the algorithm for column addition.

Unifying Ideas: Addition and subtraction; functions and relations.

Materials for Teacher: Tagboard strips like those on the pupil page.

Materials for Pupils: Cuisenaire rods.

The Lesson:

This page and pupil page 21 are designed to remind pupils of the algorithm for column addition. You may wish to have pupils work through each problem with Cuisenaire rods. For example, in exercise H a pupil would use an orange rod and a brown rod together with an orange rod and a purple rod. The brown rod and the purple rod would then be exchanged for an orange rod and a red rod. Thus, the sum is represented by three orange rods and a red rod.

Along with teaching the algorithms, we must give children an understanding of the process. By "understanding" we mean the ability to get the right answer when one forgets the algorithm. When memory fails--when a child forgets the rules--what has he to fall back on? Only on understanding. The test for understanding is just that--what can a pupil do without the algorithm?

We should see the "algorithm" as a final outcome of a learning process which involves five steps:

Step 1. Counting--The Real World of Things

If you understand what is meant by adding, subtracting, multiplying, and dividing, you can get to a correct answer to any problem by arranging objects and counting them. What is 17 + 15? Have 17 children line up on one side of the room and 15 on the other. Count them to find that 17 + 15 is 32. Tongue depressors or Popsicle sticks are more easily manipulated--so get to them as soon as possible.

Step 2. Making Pictures and Sketches

The next skill we develop is that of making pictures of things and treating the pictures as things. Abstraction has crept in. Use marks on the board instead of children or Popsicle sticks:

X X X X X X X X X X X X X X X X X (17)

. . . . . (15)

Perhaps we should notice how helpful it is to remember that 10 + 10 is 20 and start from there to count the strays on the end. Perhaps we should group the strays in 5's and remember that 5 + 5 is 10, that 10 + 10 is 20, and that 20 + 10 is 30 . . . and 2 more strays. This kind of activity leads to the next step.

Step 3. Making Simplified Sketches

It's tiring to draw so many X's and dots. So we use a shorthand notation:

(10) X X X X X X X (17)

(10) . . . . . (15)

And here is an even more abbreviated shorthand notation:

(10) (5) X X (17)

(10) (5) (15)

Step 4. Giving Arguments

Simplified sketches and tricks to avoid tedious counting lead one to use reasoning or arguments. For example, from what one knows of place value, it is clear that 17 + 15 = (10 + 7) + (10 + 5). In view of the fact that the order and grouping in an addition problem can be changed without affecting the sum, it follows that (10 + 7) + (10 + 5) = (10 + 10) + (7 + 5).

Also, 7 + 5 = 12 = 10 + 2.

So, 17 + 15 is (10 + 10) + (10 + 2)

or, (10 + 10 + 10) + 2

which, by place value, is 32.

Step 5. Using the Algorithm

Now we' re ready for the algorithm. But if we have carefully worked through each step in the development:

Things → Sketches → Brief Sketch →

Arguments → Algorithm

how much is left to do ? If pupils have not in the past been able to master the algorithms, it is probably because we have been skipping part of the development--we' ve been hurrying... and the only way to hurry to the algorithm is by using rote memory. Then, when a pupil' s memory fails, he has nothing to fall back on.

An algorithm is just a special line of argument, or a written record of manipulations with objects. If you forget that particular line of argument, use the best line of argument you can devise. If arguments are elusive, draw a brief sketch which will, of course, suggest a line of argument. If you can' t think of a brief sketch, draw a full one... if you can' t draw any sketch, then go get the Popsicle sticks!

When children falter at the algorithm step, they should be led back down toward more familiar ground--and they should keep going back until their feet find

something firm and reassuring. If you have trouble teaching the algorithms, it' s probably because you are getting there too fast--leaping over the steps we' ve outlined. If you let the program outlined above unfold, most pupils will have the algorithm before you teach it--it' s just a record of reasonable arguments--arguments that refer to a brief sketch--an abbreviation of a full sketch--a suggestion of real objects.

On pupil pages 20, 21, and 23-27, most of the activities aim toward understanding algorithms as indicated in the above discussion. Return to this series of steps frequently in analyzing the situation in your class if pupils are not making adequate progress.

You may wish to begin preparing for pupil page 20 by using strips of tagboard marked as suggested in the sketches on each page.

Hold up various combinations of pairs of such cards and have pupils tell you how many there are in all. Be sure that they notice how helpful it is to remember to count by tens and to group ones into so many tens and so many more.

(See Notes on Pupil Pages 54-65 in the Teacher' s Guide, LEVEL B (Revised) for a careful development and discussion of the algorithms for column addition and column subtraction.)

Drill

Here is a suggestion for a drill sheet to give pupils practice in hunting for tens:

1. $6 + 5 + 4 =$ ___	2. $8 + 7 + 2 =$ ___
3. $7 + 2 + 3 =$ ___	4. $6 + 2 + 8 =$ ___
5. $5 + 9 + 5 =$ ___	6. $4 + 9 + 1 =$ ___

7. 7
3
+ 5

8. 3
2
+ 7

9. 6
7
+ 3

10. 8
5
+ 2

11. 4
6
+ 8

12. 4 + 6 + 3 + 5 = ___

13. 7 + 5 + 2 + 3 = ___

14. 8
7
2
+ 1

15. 9
3
2
+ 1

16. 6
7
3
+ 2

17. 7
5
3
+ 5

18. 4
7
6
+ 3

19. 1
9
9
+ 1

Here are some stories for headline practice:

1. Reno spent 90 cents for 9 boxes of popcorn.
2. Eighteen children were divided into teams with 6 children on each team.
3. A jet pilot landed after flying 433 miles. After taking on some more passengers he took off to complete his flight plan. His plan called for a total trip of 914 miles.

Headline for the Day:

Tell a story for any of the sentences you wrote for exercises A through L on page 20.

NOTES ON PUPIL PAGE 21

Tasks: In exercises N-S pupils use the clues provided in exercise N to construct complete number sentences and partitions in each of the other exercises. In exercises T-V pupils complete each Cross Number Puzzle. In exercise W pupils use the clues provided in the first two columns, or the pattern indicators at the extreme right, to complete the table.

Purpose: To review and extend understanding of addition and of the algorithm for column addition.

Unifying Ideas: Number and counting; numeration; addition and subtraction; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

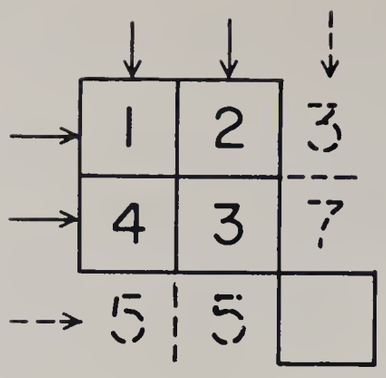
On this page pupils are exploring simplified sketches and arguments which lead to the algorithm of column addition.

Examples like those on the page should be placed on the board for discussion. Pupils should be asked why a grouping of counters in rows of 10 is better than one in which there is 8 or 11 or some number other than 10 in a row.

Cross Number Puzzles

Pupils who have used MATH WORKSHOP, LEVELS A and B, will know the secrets of the Cross Number Puzzle. Ask these pupils to keep the secrets and to look for new discoveries as examples are explored.

Put the following sketch, including the arrows, on the chalkboard:



The arrows, you will explain, tell us to add. As results of the four examples of addition indicated by the solid arrows are recorded, we are led to wonder where to record the additions indicated by the broken-line arrows.

Of course, in the one example shown above, we could write a 10 in the empty box and it would serve jointly as the sum of 3 and 7 and the sum of 5 and 5. Is this a coincidence? Most children and adults think it is, and that this outcome is dependent upon the choice of the four numbers originally written in the four cells.

Many of your children would be less surprised by the outcome if they imagined objects in the four cells - 1 dot, 2 squares, 3 circles, and 4 stars. We can sweep the objects across into piles of 3 and 7; then sweep them down into a single pile of 10. Or, we could sweep them down into piles of 5 and 5; then sweep them across into a single pile of 10.

If we started with 10 objects and swept them together, we would not expect any to appear out of thin air or to miraculously disappear. If we start with 10 objects, we will end up with 10 objects.

We might keep a record of the alternatives above in the following way:

Sweeping across and then down -

$$(1 + 2) + (4 + 3) = 3 + 7 = 10$$

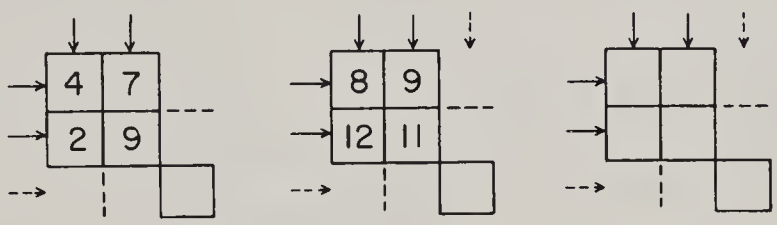
Sweeping down and then across -

$$(1 + 4) + (2 + 3) = 5 + 5 = 10$$

There is a fundamental principle in arithmetic that the grouping and the order in which numbers are added does not affect the result. In columns of addition, we often "add up" the columns and then, to check our computations, we "add down" the columns. We are confident that this is a check because we are sure that the order and grouping used when we add will not change the result.

Yet, when most children (and many adults) are first confronted with a Cross Number Puzzle, they are amazed that the box in the lower right-hand corner will suffice for recording the result of the addition in the right-hand column and the result of the addition in the bottom row.

Try the next two examples below and then write any four numbers that come to mind for a further example:



You will probably have several children in your class who can give this puzzle a real workout while trying to discover four numbers that won't work. Even after they have swept objects on the floor and visually swept designs on paper - first one way and then the other - they may not fully understand that the numbers in the cells are shorthand notation for these more concrete situations - that is, that the numbers are answers to the How many? questions we could ask if we were to construct a physical representation.

The Rearrangement Rule for Addition

In this work with Cross Number Puzzles we are trying to help pupils see examples of a basic principle in mathematics: A string of numbers to be added can be rearranged (regrouped or reordered) in any way without changing the sum. The principle depends, of course, on the fact that a display of objects can be partitioned in any manner without affecting the total number of objects in the display.

Any application of the rearrangement rule for addition can be justified by appealing to the commutative and associative principles for addition. For example, in the case of the Cross Number Puzzle given at the beginning of this discussion, the rearrangement rule tells us that

$$(1 + 2) + (4 + 3)$$

is the same number as

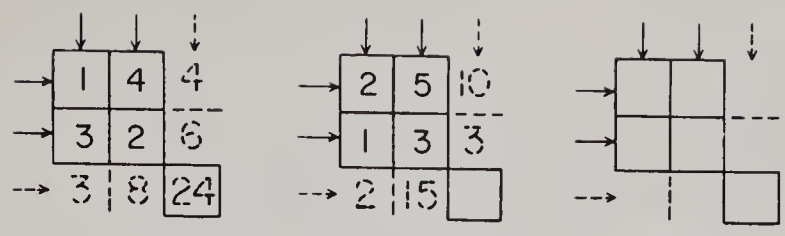
$$(1 + 4) + (2 + 3).$$

Let's see how the commutative (order) principle and the associative (grouping) principle come into the act:

$$\begin{aligned} & (1 + 2) + (4 + 3) \\ &= [(1 + 2) + 4] + 3 && \text{associativity} \\ &= [1 + (2 + 4)] + 3 && \text{associativity} \\ &= [1 + (4 + 2)] + 3 && \text{commutativity} \\ &= [(1 + 4) + 2] + 3 && \text{associativity} \\ &= (1 + 4) + (2 + 3) && \text{associativity} \end{aligned}$$

The Rearrangement Rule for Multiplication

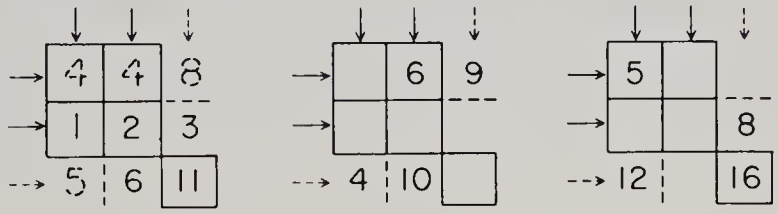
The format of the Cross Number Puzzle helps reveal that there is a rearrangement rule for multiplication, too. In the examples below, the arrows tell us to multiply. We leave to you the job of making up a third example, try any four numbers that come to mind.



Subtracting a Number Is the Inverse of Adding It

The Cross Number Puzzle may also be used to introduce the pupil to subtraction.

The agreement for the puzzle below is, as earlier, that the arrows indicate addition and that the number for the lower right-hand cell is the result of adding the four numbers in the basic four cells. Subtraction enters the picture as soon as we are faced with a problem of reconstruction, or building backwards. Here are some examples:



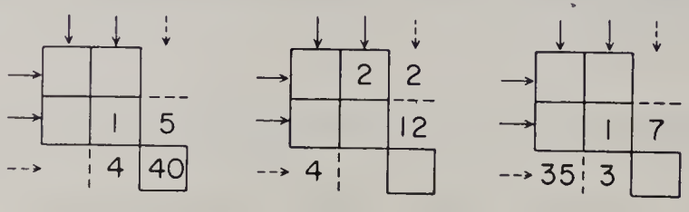
To rebuild, in the first example, you may argue as follows: I' m told that by adding 3 to some number, I get the result 11. To find this number I should "undo" the result of adding 3. In doing so, I must perform the inverse operation on 11. This inverse operation is called subtracting 3. Since 8 + 3 = 11 and there is no other number whose sum with 3 is 11, the result of subtracting 3 from 11 must be 8.

Step by step, the number that belongs in each cell in each example can be

discovered by similar reasoning. And most of the steps we take call for subtraction.

Dividing by a Number Is the Inverse of Multiplying by it

Similarly, we can reconstruct puzzles when the rule is that the arrows indicate multiplication. Examples:

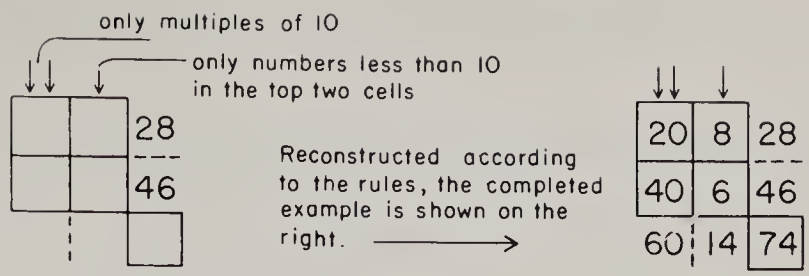


In the first example the question is asked: Multiplying what number by 5 results in 40? Or: 40 divided by 5 is what number? Etc.

The problem of reconstruction now involves division. Here we have examples of the fact that dividing by a number undoes the result of multiplying by that number.

Help with Algorithms

The addition Cross Number Puzzle will help pupils understand the steps in the algorithm for column addition. The examples below call for reconstruction according to the agreement that only multiples of 10 can be used in the column under the double arrow, and only numbers less than 10 can be used in the top two cells of the column under the single arrow. Otherwise, the normal rules of addition are used.



In following the rules, 28 and 46 are partitioned or "regrouped." This series of steps will be condensed or abbreviated and lead into the algorithm for column addition.

Drill

Consider using as a drill sheets on Cross Number Puzzles pages 24 and 25 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Bridget paid 24 cents for 3 oranges.
2. A farmer had 6 pigs in his pen and he drove in 3 more pigs.
3. Fourteen campers were joined by 18 more. All of the campers made camp by the lake.

Headline for the Day:

Anita spent 7¢ more than Jonah. What can you say about this situation? Make a table.

NOTES ON PUPIL PAGE 22

Tasks: Pupils use the clues provided in the green region at the top of the page to guide them in completing the twelve test exercises.

Purpose: To diagnose strength and weakness in addition and subtraction.

Unifying Ideas: Addition and subtraction; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Pupil page 22 is the first of 7 test pages. The others are pupil pages 39, 52, 65, 72, 88, and 99. Of course, every pupil page is a test page in a very important sense. If pupils have difficulty with any work in this book, such difficulty should be regarded as symptomatic of a need for individual help.

The particular form of the test on pupil page 22 is not designed as a comprehensive test on all work studied in pupil pages 1 through 21. Rather, it gives us a sense of the general achievement of pupils. Those who fail to get at least 80% (29 or 30 points out of 36 -- 3 points per item) correct on their own need special attention.

Prepare pupils for this page by discussing the method for marking and recording answers, using one or more examples like that at the top of the page.

Drill

Here is a suggestion for a drill sheet:

Rules:

(a) Add 2... 3, 5, 7, , , , , ,

(b) Add 3... 3, 6, 9, , , , , ,

(c) Add 5... 5, 10, , , , , , ,

(d) Add 3... 10, 13, , , , , , ,

(e) Add 4... 4, 8, , , , , , ,

(f) Add 5... 13, , , , , , , ,

(g) Add 4... 15, , , , , , , ,

(h) Add 6... 6, , , , , , , ,

(i) Add 7... 7, , , , , , , ,

(j) Add 5... 21, , , , , , , ,

(k) Add 8... 8, , , , , , , ,

(l) Add 9... 9, , , , , , , ,

(m) Add 8... 11, , , , , , , ,

Here are some stories for headline practice:

1. Gene has 50 marbles and 5 sacks. He put the same number of marbles in each of the 5 sacks.
2. Bonnie' s cousin caught 8 fish. Bonnie and her dad caught 14 fish.
3. Wednesday 545 children bought milk with their lunches. Thursday 485 children had milk with their lunches.

Headline for the Day:

Tell a story to fit the headline: $50 - 18 = ___$.

NOTES ON PUPIL PAGE 23

Tasks: Pupils use the clues and directions provided to record in the white boxes of the blue region the units digit of each sum. In the white boxes of the green region they record the tens digit of each sum. In the blue and green boxes of the white region they record the required sums. In the games at the lower right and bottom of the page pupils use the clues to complete the table.

Purpose: To focus attention on the steps in the algorithm for column addition.

Unifying Ideas: Numeration; addition and subtraction; functions and relations.

Materials for Teacher: Counting Board and washers.

Materials for Pupils: None.

The Lesson:

On this page, pupils examine 18 problems in three ways:

First, they focus their attention on the units place of the sum.

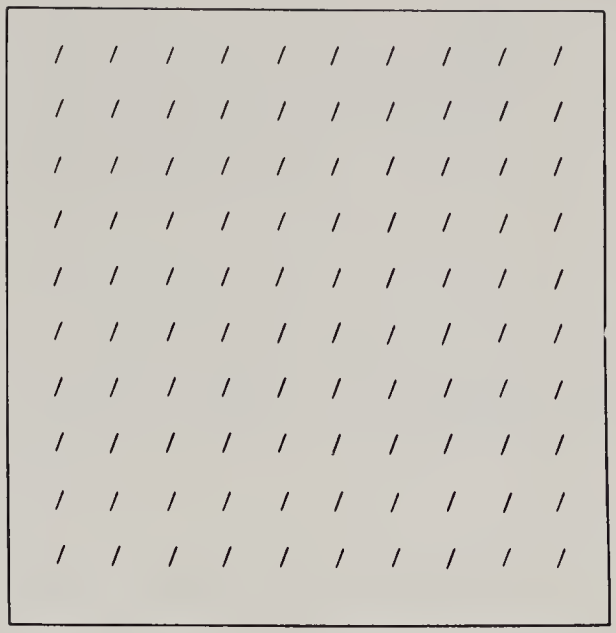
Second, they focus their attention on the tens place of the sum.

Finally, they work the problem completely.

This attention to the algorithm one step at a time increases a pupil's insight into the arguments behind the addition algorithm.

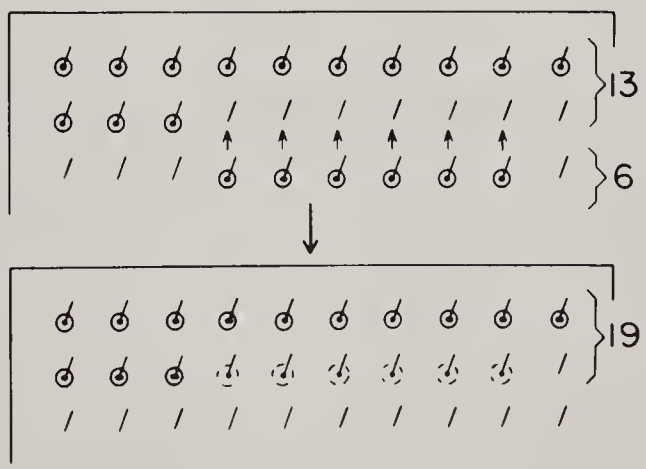
The Counting Board

Now is a good time to bring in the Counting Board. This versatile device can be made from a square piece of wood with a 10-by-10 arrangement of 100 nails driven in part way, and a box of washers which can be obtained at any hardware store. Use finishing nails -- the kind without heads -- so that the washers will slip over them easily.

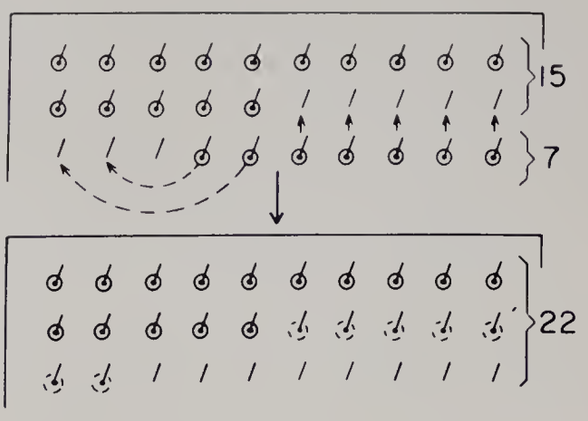


A few of the many uses of the Counting Board are shown in the following examples:

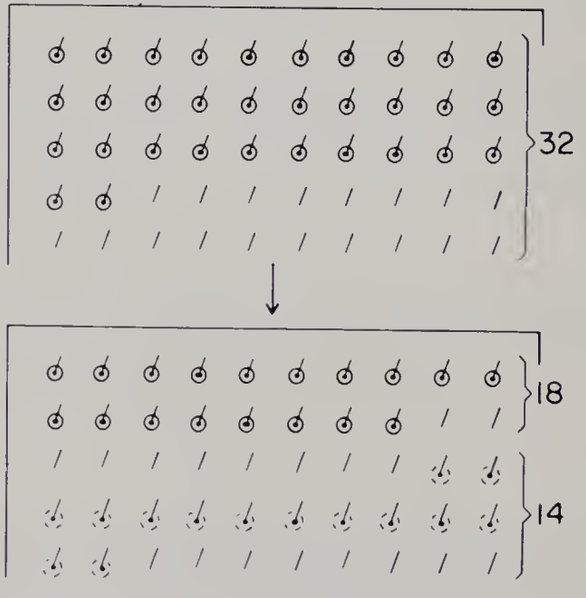
13 + 6 = 19 15 + 7 = 22 18 + 14 = 32



13 + 6 = 19 = 6 + 13



$15 + 7 = 22 = 7 + 15$



$18 + 14 = 32 = 14 + 18$

Any of the demonstrations can be regarded as interpreting any of the members of a fact team, such as:

$15 + 7 = 22$ $\qquad$ $7 + 15 = 22$

$22 - 7 = 15$ $\qquad$ $22 - 15 = 7$

You will find also the Counting Board to be very helpful when you are working with multiplication and division.

Drill

Consider using as a drill sheet page 17 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Roy had 16 pieces of candy. He gave some to Pete. Each boy now has the same number of pieces of candy.

   $16 - ___ = ___$ $\qquad$ $16 \div 2 = 8$

2. Mr. Gilgoyle's third grade class had 99 cookies for their class party. After the party 17 cookies were left.

3. Petulla guessed there were 200 people at the boys' basketball game. There were actually 188 people at the game.

Headline for the Day:

Kerby spent 3¢ less than Ingrid. Make a table to keep a record of what you can say about this situation.

NOTES ON PUPIL PAGE 24

Tasks: Pupils use the clues provided to record in the tinted boxes the required differences and, in F-N, to reconstruct the subtraction problems.

Purpose: To give practice with representations which lead to the algorithm for column subtraction.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Graph board; colored chalk.

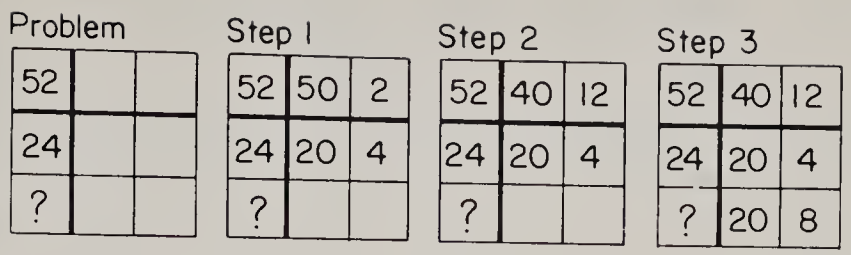
Materials for Pupils: Cuisenaire rods; graph paper.

The Lesson:

An understanding of the algorithm for column subtraction is easily developed by using graph paper as illustrated on pupil pages 24 and 25. It may also be developed with Cuisenaire rods. For example, consider the problem of subtracting 24 from 52. We represent 52 by 5 orange rods and 1 red rod. Our job is to remove 2 orange rods and a purple rod from this representation of 52. In order to do this, one of the orange rods must be exchanged for, say, a brown rod and a red rod. Now, 52 is represented by 4 orange rods, 1 brown rod, and 2 red rods. Remove 2 of the orange rods and the 2 red rods. We are left with 2 orange rods and 1 brown rod. Thus, the result of subtracting 24 from 52 is 28.

Cross Number Puzzles

The subtraction algorithm can also be developed through Cross Number Puzzles as we did in LEVEL B.



The puzzle is oriented so that the total-box is in the upper left-hand corner. The job is now one of reconstructing the puzzle in order to find the number for the box in the lower left-hand corner. We partition 52 and 24 as shown in Step 1. This is not a useful partition because there is no whole number whose sum with 4 is 2. Step 2 shows a more useful partition of 52. The number for the lower left-hand cell is the sum of 20 and 8.

Drill

Here is a suggestion for a drill sheet:

1. 50 + 70	2. 80 + 30	3. 20 + 30
4. 60 + 70	5. 90 + 80	6. 40 + 70
7. 16 + 32	8. 36 + 41	9. 32 + 55
10. 43 + 31	11. 83 + 15	12. 50 + 40

13. 76 + 52	14. 54 + 82	15. 96 + 42
16. 83 + 83	17. 50 + 68	18. 42 + 53

Here are some stories for headline practice:

1. Lisa paid 40 cents for 8 apples.
2. Manfred had 20 cents. His dad gave him 70 cents for his weekly allowance.
3. Melissa counted 51 boxcars, 23 flatcars, and 14 gondolas on the freight train.

Headline for the Day:

Tell a story to fit this headline: $126 - 39 = ____$.

NOTES ON PUPIL PAGE 25

Tasks: Pupils use the clues provided to record in the tinted boxes the required differences.

Purpose: To further explore representations which lead to the algorithm for column subtraction, and to provide practice in using the algorithm.

Unifying Ideas: Number and counting; addition and subtraction.

Materials for Teacher: Graph board.

Materials for Pupils: Cuisenaire rods; graph paper.

The Lesson:

Although there should be little need to do so, you may wish to run through the steps used in column subtraction.

Example 1.

$$\begin{array}{r} 59 \\ -\,17 \\ \hline 2 \end{array}$$

First, subtract 7 from 9. The answer is 2. Write a 2 in the units place.

$$\begin{array}{r} 59 \\ -\,17 \\ \hline 42 \end{array}$$

Next, subtract 10 from 50. The answer is 40. Write a 4 in the tens place.

Example 2.

$$\begin{array}{r} 53 \\ -\,17 \\ \hline \end{array}$$

First, subtract 7 from 3. This cannot be done. So, think of 53 as 40 + 13. Show this by crossing out the 5, writing a 4, and a 1. (Or, cross out the 3 and write a 13.)

$$\begin{array}{r} {}^{4}\cancel{5}^{1}3 \\ -\,17 \\ \hline 6 \end{array}$$

Now, subtract 7 from 13. The answer is 6. Write a 6 in the units place.

$$\begin{array}{r} {}^{4}\cancel{5}^{1}3 \\ -\,17 \\ \hline 36 \end{array}$$

Next, subtract 10 from 40. The answer is 30. Write a 3 in the tens place.

Some pupils may feel more comfortable if they put in the "scratch marks." Others will not need to do this at all. We prefer not to establish a uniform policy in such matters.

Drill

Consider using as a drill sheet page 23 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Claire put 7 pencils in each of nine boxes.
2. Nina's mother made 12 sandwiches for Nina and her 3 friends to take on their hike. They divided the 12 sandwiches equally among themselves.
3. Pauline's third grade class made a patchwork quilt. There were 45 green squares and the same number of blue squares.

Headline for the Day:

Tell a story to fit this headline: $27 - 27 = \underline{\quad}$.

NOTES ON PUPIL PAGE 26

Tasks: Pupils use the clues and directions provided to record in the white boxes of the green region the units digit of each result. In the white boxes of the blue region they record the tens digit of the result. In the blue and green boxes of the white region pupils record the required results. For the What' s My Rule? games pupils use the clues provided to complete the tables.

Purpose: To focus attention on the steps in the algorithm for column subtraction.

Unifying Ideas: Numeration; addition and subtraction; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

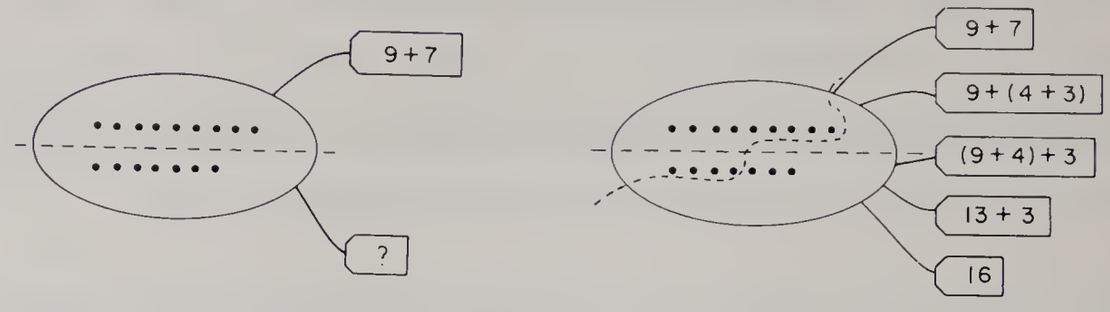
The exercises on this page are designed to allow pupils to focus their attention on the steps involved in finding the units digit and the tens digit in the answer to a problem involving the subtraction of a 1-digit or a 2-digit number from a 2-digit number.

Partitioning--The Basis for the Algorithms

Let' s pretend that we have forgotten the basic fact that 9 + 7 is 16. How could we rediscover it? Our basic procedure is to take a display of 9 objects and a display of 7 objects none of which is in the display of 9 objects and combine the two displays. The sum of 9 and 7 is the number of objects in the combined display.

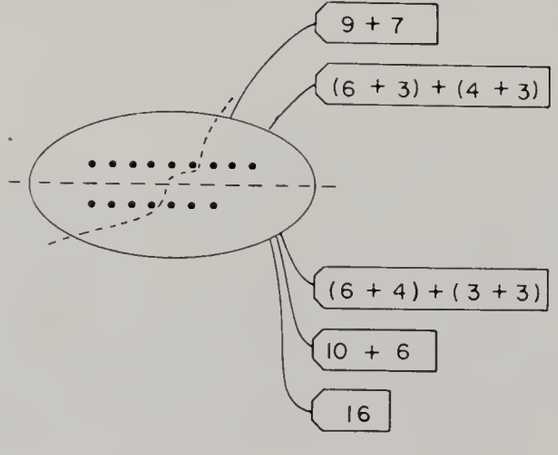
We could determine the sum by straightforward one-by-one counting. We might avoid the counting by using some basic addition facts already known to us. For example, if we had memorized the facts that 7 is 4 + 3, 9 + 4 is 13, and

13 + 3 is 16, our problem would be solved. For in this case we would partition the combined display as follows:

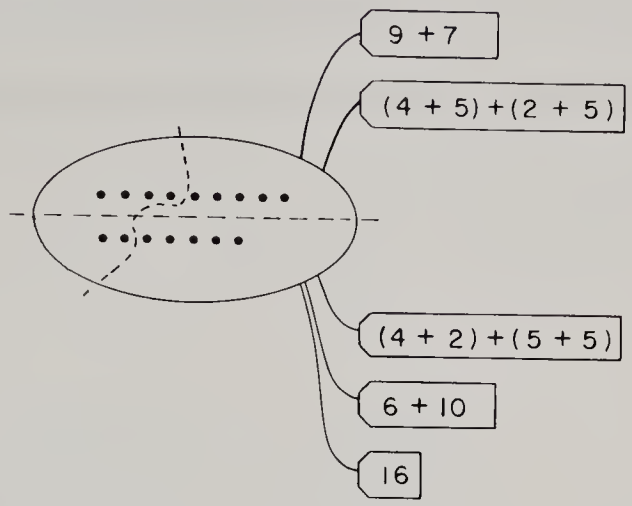


Our acquaintance with the world of physical objects assures us that this partitioning will not change the sum.

Of course there are many partitionings we might use. Our selection depends on which addition facts we know. For example, suppose that we know that 9 is 6 + 3, 7 is 4 + 3, 6 + 4 is 10, 3 + 3 is 6, and 10 + 6 is 16. With this knowledge at our disposal, we would partition the combined display as follows:



Another way of discovering that 9 + 7 is 16 depends on the knowledge that 9 is 4 + 5, 7 is 2 + 5, 4 + 2 is 6, 5 + 5 is 10, and 6 + 10 is 16:



Partitioning and Positional Notation

Now, of course, it is highly unlikely that someone who had memorized all the addition facts mentioned above would have failed to memorize the fact that 9 + 7 is 16. The purpose of the foregoing discussion is to illustrate the general procedure that partitioning does not affect the total. The general procedure becomes useful when you can choose a partitioning which allows you to make use of memorized facts.

Consider how we might find the sum of 46 and 32. One could not be expected to have memorized this fact and it would be quite tedious to try to count, one-by-one, the combination of a display of 46 objects and a display of 32 objects.

Instead, one starts calling upon certain basic addition facts and his knowledge of positional notation. Here are the facts:

(1) $46 = 40 + 6$ [positional notation]

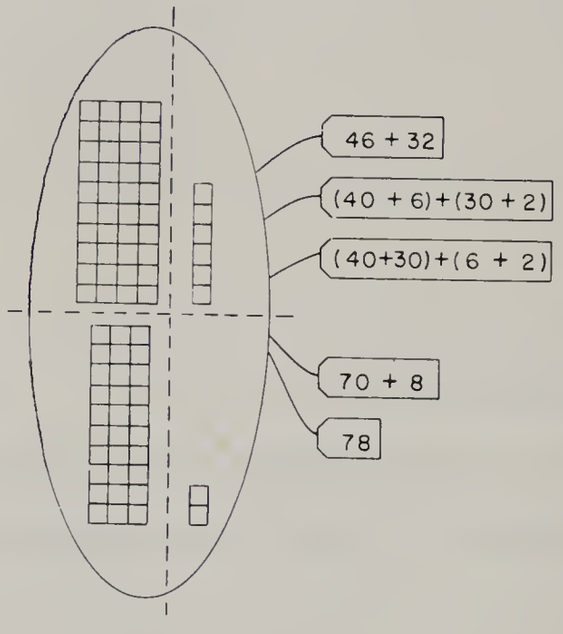
(2) $32 = 30 + 2$ [positional notation]

(3) $40 + 30 = 70$ [positional notation and basic fact]

(4) $6 + 2 = 78$ [basic fact]

(5) $70 + 8 = 78$ [positional notation]

With these five facts at hand, one partitions the combined display as follows:



We can keep track of these steps in more abstract form as follows:

Problem: $46 + 32 = ?$

Step 1: Get ready for a partitioning.

		46
		32
		?

Step 2: Partition in accordance with positional notation. (Facts (1) and (2).)

40	6	46
30	2	32
		?

Step 3: Avoid the counting by using basic memorized facts and positional notation. (Facts (3), (4), and (5).)

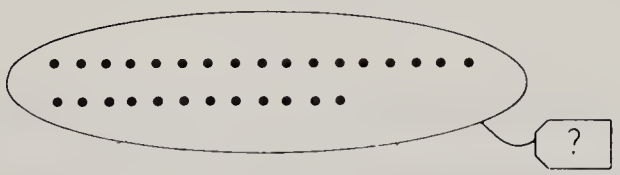
40	6	46
30	2	32
70	8	78

Our knowledge that no rearrangement of a physical display will change the count is the basis for asserting that the sum of 70 and 8 is the sum of 46 and 32.

Finally, positional notation is such a powerful device that one need not be conscious of choosing a partitioning. The choice is made as soon as you decide to line up the places and add in columns.

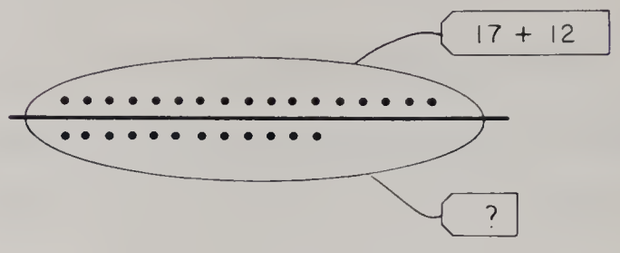
An Analysis of Addition without Exchanging

Let' s analyze the job of finding the sum of 17 and 12. In physical terms this amounts to finding the number of objects in a combined display of two nonoverlapping displays--a display of 17 objects and a display of 12 objects. Here is a picture of such a combined display:



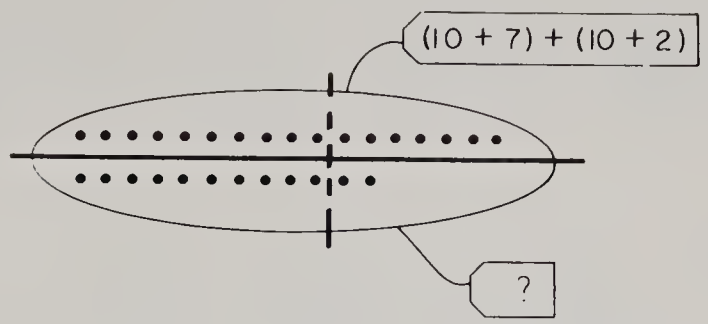
The job of finding the total really amounts to reporting the number in such a way that a person receiving the report will get an accurate "picture" of the numerosity of the combined display. We ordinarily assume that such a person has an excellent acquaintance with the numbers from 0 through 10. So, we ordinarily give our report in terms of a partition each of whose parts consists of no more than 10 entities. In other words, we ordinarily give our report in terms of a decimal partitioning. (If, however, our respondent were limited in his acquaintance to just the numbers from 0 through 5, we would give our report in terms of a quinary partitioning.)

(One of the things we know immediately about the combined display is that it can be partitioned into a

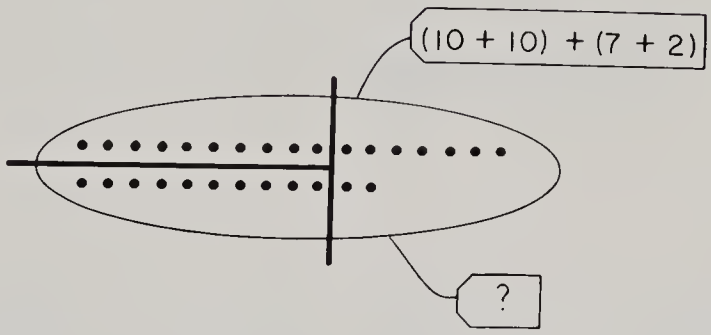


17-group and a 12-group. If our respondent were well acquainted with the numbers 17 and 12, all we would need to report to him is that the combined display could be partitioned into one 17-group and one 12-group. This report would give him a clear picture of the numerosity of the combined display. He would feel perfectly comfortable with the symbol "17 + 12" as a name for the sum. But, we are assuming that our respondent feels comfortable only with reports which refer to groups which consist of at most ten objects. So he would not feel comfortable with "17 + 12" as a report.)

Now how do we go about obtaining a decimal partitioning of the combined display? We start with the knowledge that one partitioning is 17 + 12. Because of our understanding of positional notation and because we know that a decimal partitioning was used in reporting the numbers for the given displays, we know that one of the given displays can be partitioned into a 10-group and seven 1-groups, and the other into a 10-group and two 1-groups. We have



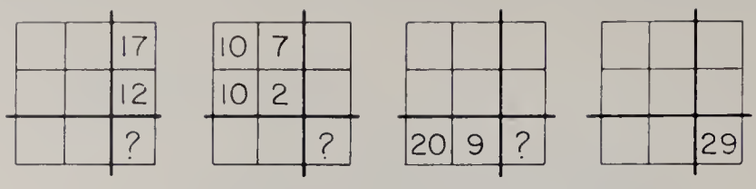
discovered that the combined display can be partitioned into two 10-groups and one group consisting of 7 + 2 objects.



Since 7 + 2 is less than 10, the partitioning job is finished.

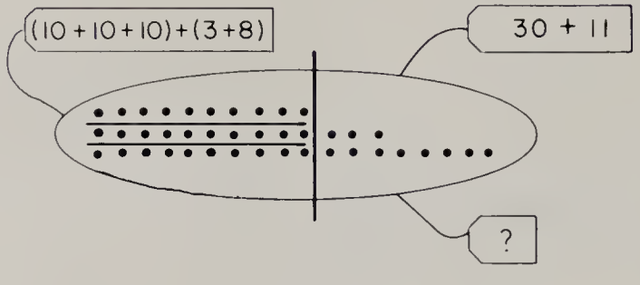
We are now ready to give the report. Because we have memorized certain basic facts we know that 7 + 2 = 9. We can tell our decimal-minded respondent that the combined display can be partitioned into two 10-groups and nine 1-groups. For short, we simply tell him that the combined display consists of 29 members. [The symbol "29" means nothing more than two 10-groups and nine 1-groups.]

The Cross Number Puzzle form shows the steps very neatly:

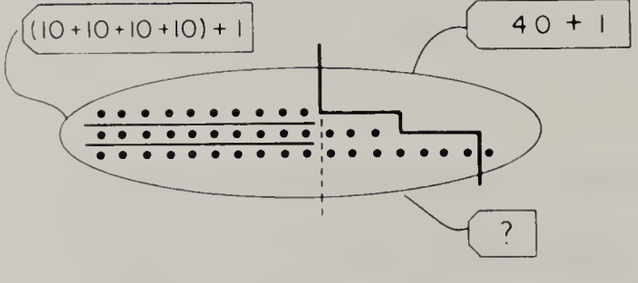


Addition with Exchanging

Now let' s consider the job of finding the sum of 30 and 11. We shall have to do more work to achieve the partition we are seeking. For example, here might be one of the steps:



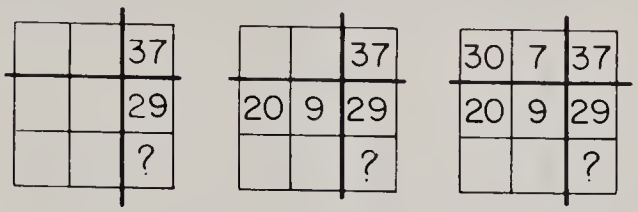
Since 3 + 8 is greater than 10, our job is not finished. Here is the finished job:



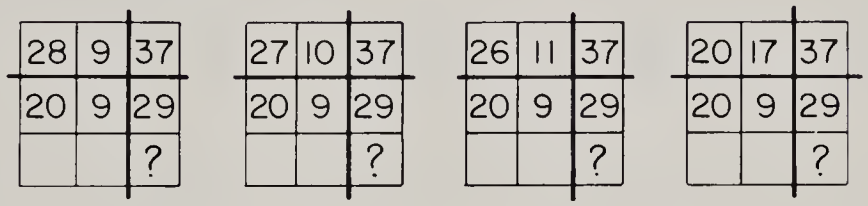
The regrouping from the partition 30 + 11 to the partition 40 + 1 is commonly called carrying. But this term is certainly not suggested by any of the work we have done. It might be better to use the word exchanging because it is more suggestive of what is done.

An Analysis of Subtraction

Let' s consider the problem of subtracting 29 from 37. A pupil can solve this problem by reconstructing an addition Cross Number Puzzle. A pupil might first reconstruct as follows:



At this stage, he realizes that there is no whole number he can add 9 to in order to get the sum 7. Given complete freedom, he might decide to change his partition of 29 from 20 + 9 to 22 + 7. But if he wishes to stick with a decimal partitioning of 29, he would not do this. So he must change the partition of 37. Any of the following could help:



However, only the last partition capitalizes fully on positional notation. (Otherwise, one would be hard-pressed to defend it over the first or second.)

The regrouping from the partition 30 + 7 to the partition 20 + 17 is commonly called borrowing. Clearly, this term is not suggested by any of the work we have done. As with carrying, you may wish to use the word exchanging instead.

Drill

Here is a suggestion for a drill sheet:

	I	II	III	IV	V	VI
	10	10	9	20	4	37
A	+ 0	+ 7	+ 10	+ 6	+ 30	+ 20
	2	4	6	2	1	5
B	3	2	3	0	7	5
	+ 5	+ 4	+ 1	+ 8	+ 2	+ 2
	8	3	1	6	3	2
C	2	5	9	4	9	5
	+ 7	+ 5	+ 4	+ 6	+ 7	+ 8
	6	12	7	5	1	6
D	10	8	3	20	40	4
	+ 4	+ 0	+ 20	+ 5	+ 9	+ 15
	1	8	17	24	7	6
E	19	5	3	1	8	34
	+ 6	+ 15	+ 6	+ 6	+ 13	+ 9
	3	6	9	3	9	5
F	7	3	6	7	4	5
	4	2	2	2	1	7
	+ 4	+ 8	+ 1	+ 8	+ 6	+ 4

	17	12	15	13	15	21
G	3	9	5	17	2	19
	+ 10	+ 8	+ 17	+ 6	+ 15	+ 5

Here are some stories for headline practice:

1. Aldo bought ten 5-cent stamps.
2. Simeon' s third grade class had an art show. The girls displayed 37 paintings. The boys displayed 25 paintings.
3. The three third grade classes in Bunny' s school put on a play together. There were 35 children in one class, 29 in another class, and 31 in the other class.

Headline for the Day:

Nevin spent a penny more than Clara. Make a table to keep a record of what you can say about this situation.

NOTES ON PUPIL PAGE 27

Tasks: Pupils use the clues and directions provided to record in the white boxes of the blue region the units digit of the required sum or difference. In the white boxes of the green region pupils record the tens digit of the required sum or difference. In the blue and green boxes of the white region they record the required sums or differences. For the games at the lower right and bottom of the page, pupils use the clues provided to complete the tables.

Purpose: To further develop efficient procedures for adding and subtracting 2-digit numbers.

Unifying Ideas: Numeration; addition and subtraction; functions and relations.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets.

The Lesson:

Addition and subtraction problems are mixed on this page but again the pupils are asked to focus their attention on the units place and on the tens place before tackling the problem as a whole.

Partitioning

As we have seen in Notes on Pupil Page 26, the algorithm for column addition of 2-digit numbers is nothing more than a written record of a decimal partitioning of a display of objects.

Let' s look at the situation a little more abstractly. Each whole number between 9 and 100 is the sum of some multiple of 10 and some whole number less than 10. The decimal notation for the given number tells you precisely which multiple of 10 and which whole number less than 10.

Consider the problem of finding the sum of 73 and 28.

$73 + 28 = (70 + 3) + (20 + 8)$ [positional notation]

$= (70 + 20) + (3 + 8)$ [rearrangement rule]

$= 90 + 11$ [basic facts]

$= 90 + (10 + 1)$ [positional notation]

$= (90 + 10) + 1$ [rearrangement rule]

$= 100 + 1$ [basic fact]

$= 101$ [positional notation]

Now, when we use the algorithm for column addition, we certainly do not write down all of this information. We use shortcuts, and the shortcuts can be justified in the same way as are the steps above.

Consider:

$$\begin{array}{r} 73 \\ +\ 28 \\ \hline \end{array}$$

If we start our "figuring" by saying to ourselves: 3 + 8 is ___, we certainly have already partitioned 73 and 28 in our minds, for how else did we pick 3 and 8?

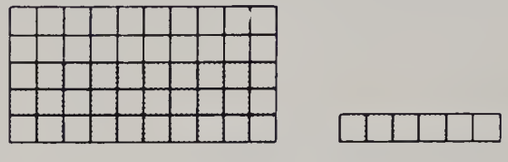
Next, we say to ourselves: 3 + 8 is 11; write a 1 in the units place and remember 10. We have partitioned 11 into 10 + 1. We also know that the rest of the problem will not involve groups smaller than 10; so, we are sure of the units digit in the sum. (It is this certainty that leads us to use a right-to-left algorithm. If we were merely estimating the sum, we would just work with the tens digits.)

Then, we say to ourselves: 7 + 2 is 9 (this is short for 70 + 20 is 90.) Then: 9 + 1 is 10 (or: 90 + 10 is 100). We record the result by writing a 0 in the tens place and a 1 in the hundreds place. All of this can be done with understanding only by someone with a good grasp on the notion of decimal notation.

Finally, the fact that we can construct a name for the sum by adding in columns depends not only on decimal notation but also upon the rearrangement principle for addition which, in turn, depends upon the fact that a physical rearrangement of the objects in a display doesn' t change the number of objects.

This last remark may appear to deal with obvious matters. But, to see that there is some substance to it, consider the problem of justifying the algorithm for column subtraction. In terms of displays of objects, subtracting in columns makes good sense:

$$\begin{array}{r} 56 \\ -\ 34 \\ \hline \end{array}$$



But, from an abstract point of view, it isn' t at all obvious that

$$(50 + 6) - (30 + 4) = (50 - 30) + (6 - 4).$$

To justify this, we make use of the fact that adding a number and subtracting that number are inverses of each other and of the rearrangement rule for addition. We start by adding $30 + 4$ to $(50 - 30) + (6 - 4)$.

$$\begin{aligned} & [(50 - 30) + (6 - 4)] + (30 + 4) \\ = \ & [(50 - 30) + 30] + [(6 - 4) + 4] \qquad \} \text{ [rearrangement]} \\ = \ & 50 + 6 \qquad \} \text{ adding a number is the inverse of subtracting it} \end{aligned}$$

So,

$$50 + 6 = [(50 - 30) + (6 - 4)] + (30 + 4).$$

Since subtracting a number is the inverse of adding it, the last equation implies that

$$(50 + 6) - (30 + 4) = (50 - 30) + (6 - 4).$$

And this is what we wished to establish.

It seems clear from the sophistication of the preceding demonstration that any attempt to justify the algorithm for column subtraction by referring to properties of numbers would be out of place in a primary grade classroom. The most practical thing to do is to refer to manipulations with physical objects or to a reconstructed Cross Number Puzzle, which is merely a record of such manipulations:

56	50	6
34	30	4
?		

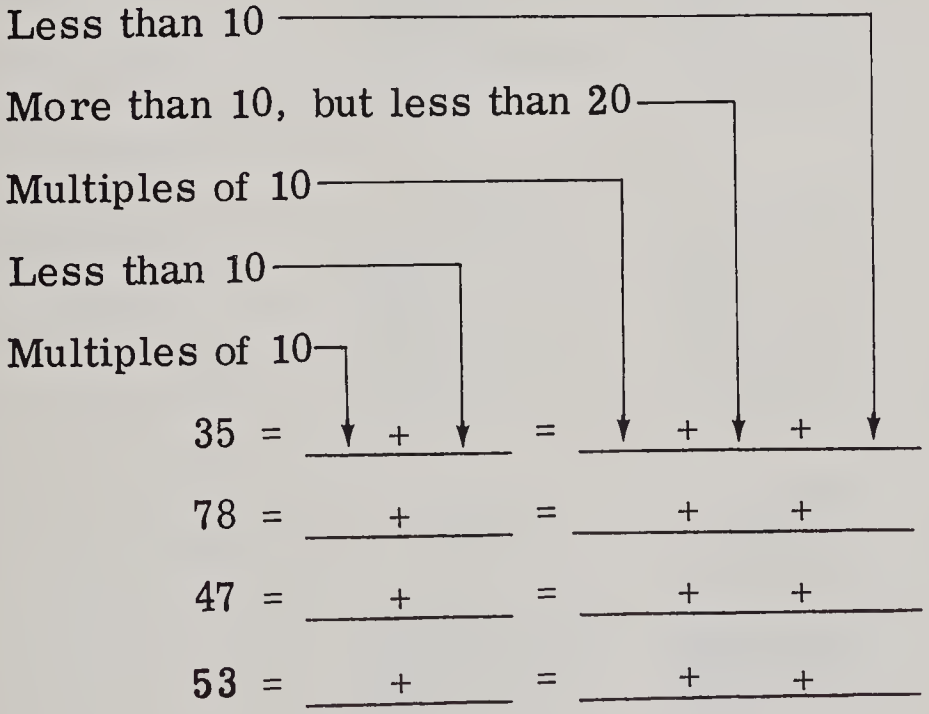
	50	6
	30	4
?	20	2

22	20	2

Partitioning Games

The idea of choosing a partitioning according to specified rules can be introduced to pupils by the following kind of board work:

Game 1



Game 2

Odd or even number less than 10
Even number less than 10
Odd number less than 10
Multiples of 10

18 = 10 + 5 + 2 + 1

91 = 80 + 7 + 4 + 0

Game 3

Using Cross Number Puzzles, the rules are that the top two cells in Column A are reserved for multiples of 10 while the top two cells in column B are reserved for numbers less than 10.

A	B	
		12
10	0	10

A	B	
	0	10
		16

A	B	
	0	20
		12

A	B	
	0	80
	0	10

A	B	
		19
0		3

A	B	
0		7
		19

A	B	
		24
		8

A	B	
		85
		5

A	B	
		17
		48

A	B	
		8
		63

A	B	
		19
		43

A	B	
		47
		60

Drill

Consider using as a drill sheet page 16 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Carlos bought seven 5-cent stamps.
2. Judy spent 23 days of her summer vacation at Granny' s. She had 37 days left in the summer holidays.
3. Danny lives near the railroad station. One week he counted 35 northbound and 27 southbound trains stopping at the station.

Headline for the Day:

Keith spent a dime more than Stella. Make a table to keep a record of what you can say about this situation.

NOTES ON PUPIL PAGE 28

Tasks: For the first four exercises pupils record in the white boxes the number of slots in each tinted card; and in the tinted boxes they record the product of the numbers of slots in cards of adjacent pairs. For the lower portion of the page pupils use the information provided by the pictures and the described operations to compute the missing information for each table. Note the "fewest units" restrictions.

Purpose: To review and extend knowledge of multiplication and to provide a use for this knowledge in several applications.

Unifying Ideas: Multiplication and division.

Materials for Teacher: Demonstration-size slotted cards as described in Notes on Pupil Page 8; 4 dimes and 16 pennies; 4 egg cartons and 64 marbles or other objects to represent eggs.

Materials for Pupils: None.

The Lesson:

Note that all of the computation problems on pupil page 28 are presented in a concrete setting so that pupils must decide which data are relevant data and which operations to use in order to do the exercises. This is particularly true of problems in V and VI; these exercises are in a format which is used heavily throughout LEVEL C.

In some elementary mathematics programs the "4 times as much" problem in V would be given as a straightforward multiplication problem:

$$\begin{array}{r} 14 \\ \times\ 4 \\ \hline \end{array}$$

and such a problem would be delayed until the algorithm for multiplying a 2-digit number by a 1-digit number had been taught.

In other programs the problem might be posed in the form of a story:

Bill had 14¢. John had four times as much. How much did John have?

The format we have adopted allows a pupil to derive the same benefit he would get from the story problem but it doesn' t burden him with reading. Also, in a very small space we present four such story problems, and we make it possible for pupils to observe certain relationships among the problems. For example, when he has solved the "twice as much" problem, he can use the answer as a datum for solving the "4 times as much" problem. (4 times as much is twice as much as twice as much.)

Finally, the format we have used in V makes it possible for the pupil to find answers without knowing the standard algorithms. Clearly, an amount 4 times as much as 1 dime and 4 pennies can be made with 4 dimes and 16 pennies. (This involves two of the steps in the standard algorithm for multiplication.) But, the pupil is also asked to use the smallest possible number of dimes and pennies to make up this same amount of money. (To find this out, he goes through another step in the standard algorithm.)

Fewest Units

In case pupils need help with V and VI, use the actual objects as a source of explanation. You may also wish to prepare for V and VI by working with the following exercises at the chalkboard:

With only nickels and pennies available, how can I make the listed amounts, using the fewest possible coins?

Amount	4 ¢	5 ¢	6 ¢.	10 ¢	15 ¢	18 ¢
Nickels	0	1				
Pennies	4	0				

Represent the time periods using the fewest possible units.

Period	3 days	7 days	13 days	14 days	23 days	
Weeks	0	1				3
Days	3	0				4

Represent the lengths using the fewest possible units.

Length	5 in.	12 in.	13 in.	27 in.	38 in.	
Feet	0	1				3
Inches	5	0				3

Drill

Here is a suggestion for a drill sheet on subtraction:

	I	II	III	IV						
	8	12	13		10	19	3	20	31	72
A	− 3	− 3	− 3	− 3						
	10	19	32		28	40	23	51	34	100
B	− 5	− 5	− 5	− 5						

C

12	15	20
− 4	− 4	− 4

− 4	29	32	13	41	83	124

D

15	21	89
− 7	− 7	− 7

− 7	47	30	54	72	36	103

E

14	22	30
− 8	− 8	− 8

− 8	43	26	51	35	99	117

F

15	20	34
− 6	− 6	− 6

− 6	23	87	41	59	136	92

G

14	18	30
− 9	− 9	− 9

− 9	21	45	33	96	62	107

H

19	20	27
− 12	− 12	− 12

− 12	18	24	30	22	70	51

I

Rule: Subtract 7.

89, 82, , , 61, , , , , ,

Here are some stories for headline practice:

1. Pat paid 4 dimes for a toy. Her cousin paid 8 nickels for a toy just like the one Pat bought.
2. Allison had a collection of sea shells. Vince brought his collection of 185 shells over to Allison' s house. Together they had 335 sea shells.

3. Sophia' s class put on a puppet show. The names of 96 boys and 79 girls were on the visitors list.

Headline for the Day:

Cecil spent 43¢ at the candy store. He gave the clerk a dollar bill. The clerk gave Cecil the correct change using the fewest coins he could. What were the coins Cecil received ? How many of each kind of coin did he receive?

NOTES ON PUPIL PAGE 29

Tasks: Pupils record in the blue boxes the product of the numbers of lines pictured on pairs of adjacent cards. At the bottom of the page pupils count or use short-cuts to counting to record in the blanks the numbers of nails and chairs pictured.

Purpose: To further extend knowledge and understanding of multiplication.

Unifying Ideas: Structure; multiplication and division.

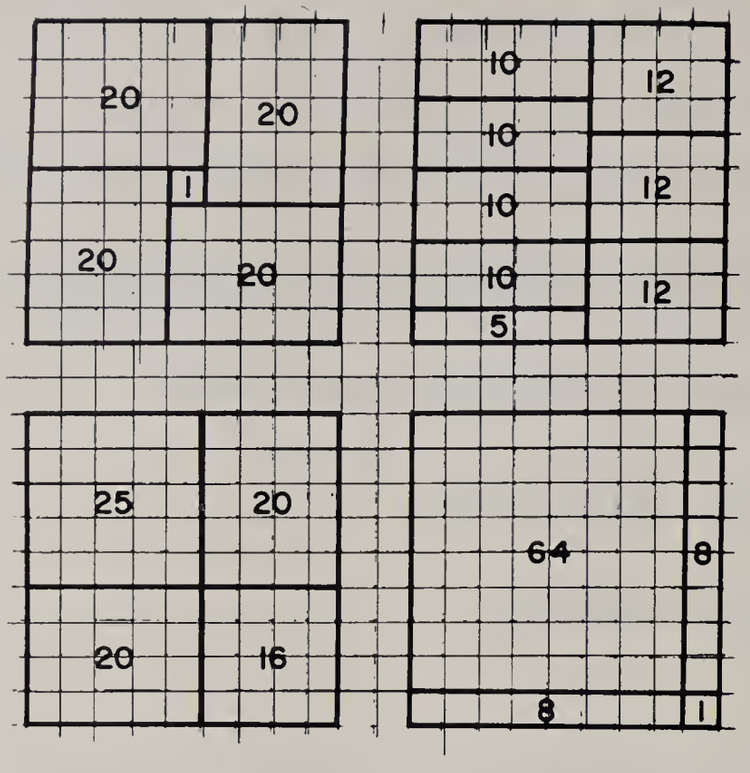
Materials for Teacher: Graph board.

Materials for Pupils: Graph paper.

The Lesson:

Encourage pupils to use their own diagrams to discover multiplication facts. Show them shortcuts to counting. For example, in finding the product of 9 by 9, a pupil might sensibly draw a rectangle on graph paper and then count the number of unit squares, one by one. This approach through counting should not be tolerated at this level. You should take an active part in encouraging the use of short-cuts by praising those pupils who use them and by demonstrating their use to pupils who may be insecure.

The basic idea in avoiding the counting of rectangular arrays is to partition them into smaller rectangular arrays whose counts have been memorized. The total is then obtained by addition. Here are some examples of such partitionings:

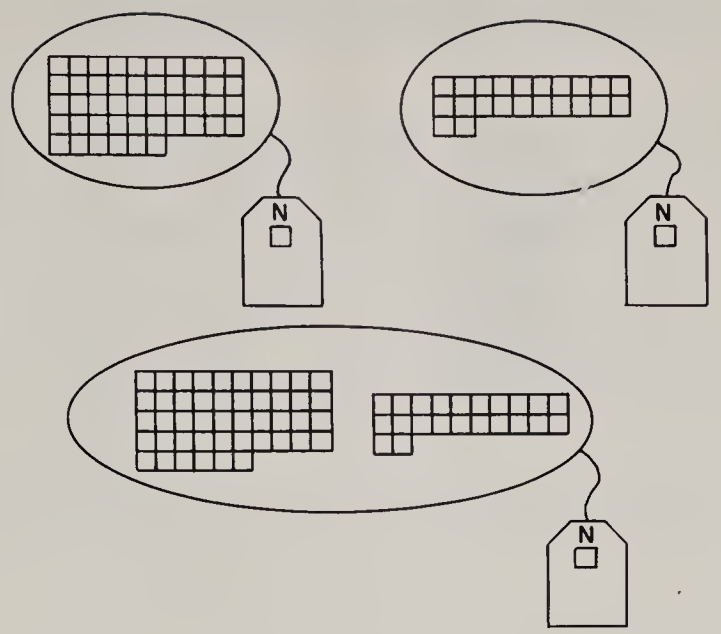


The partitioning indicated by drawing lines such as those in the bottom pair of pictures provides a basis for explaining the fact that multiplication distributes over addition. It is also the basic idea used in explaining the algorithm for long multiplication treated later in LEVEL C.

Drill

Here are some developmental exercises for a Ditto sheet on addition and subtraction:

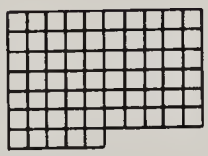
1.



2.

35	58	17	27
+ 24	+ 21	+ 42	+ 41

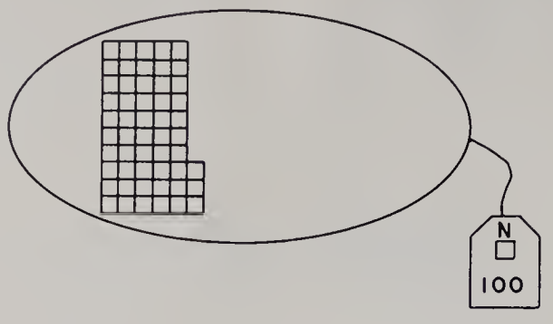
3. Shade 32 little squares and tell how many are left unshaded.



4.

59	38	27	85
− 21	− 22	− 21	− 35

5. Draw more little squares to make 100 little



squares in all. How many did you need to draw?

6.

63	63	63	63
+ □	+ □	+ □	+ □
70	80	90	100

Here are some stories for headline practice:

1. Jonas used 49 logs from a pile of 66 logs to build the floor of his log cabin.
2. Clarissa and Sam baked cookies for the cookie sale. Clarissa baked 87 cookies. Sam baked 63 cookies.
3. Mr. Block can drive his car about 13 miles on one gallon of gas. He used 5 gallons of gas to make a trip to the city.

Headline for the Day:

Lance spent 4¢ less than Tania. Make a table to keep a record of what you can say about this situation.

NOTES ON PUPIL PAGE 30

Tasks: Pupils supply the missing sums and scores in accordance with the stated conditions. They may have to do a bit of exploring for exercise E. The objective here is to arrange the numbers so that no two of the eight rows, columns and diagonals have the same sum. There are many such arrangements.

Purpose: To explore Magic Squares while practicing addition and subtraction skills.

Unifying Ideas: Addition and subtraction.

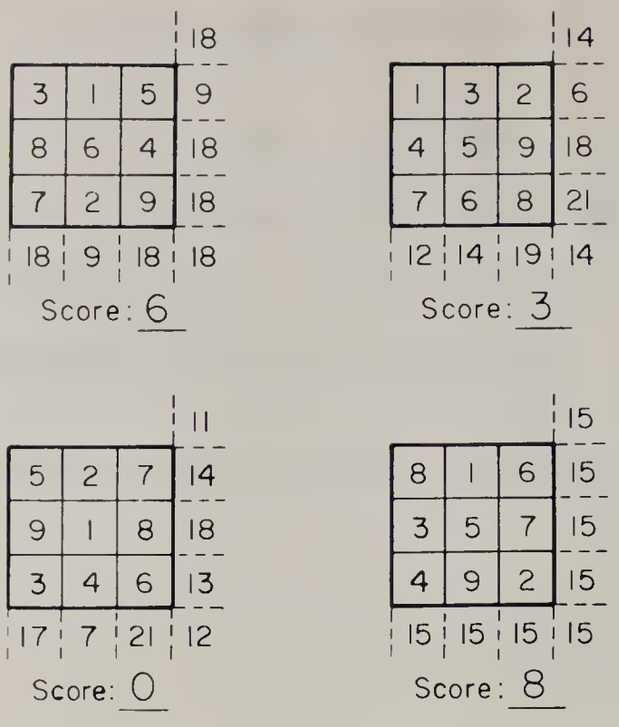
Materials for Teacher: Ditto master.

Materials for Pupils: Ditto sheets.

The Lesson:

In Magic Squares the sum for each row, column and diagonal is the same number. Magic Squares are interesting to pupils, offer opportunities for drill, and their structure invites further exploration. On pupil page 30 (and in F, G, and H on pupil page 31), Numo makes up a game in which the score for a completed square is the total number of rows, columns, and diagonals having the sum 15.

An interesting alternative approach to the subject would be to propose a game in which the score for a completed square is the largest number of rows, columns, and diagonals having the same sum. Here are four completed squares and their scores:



Pupils might then be challenged to find a square with a score of 8.

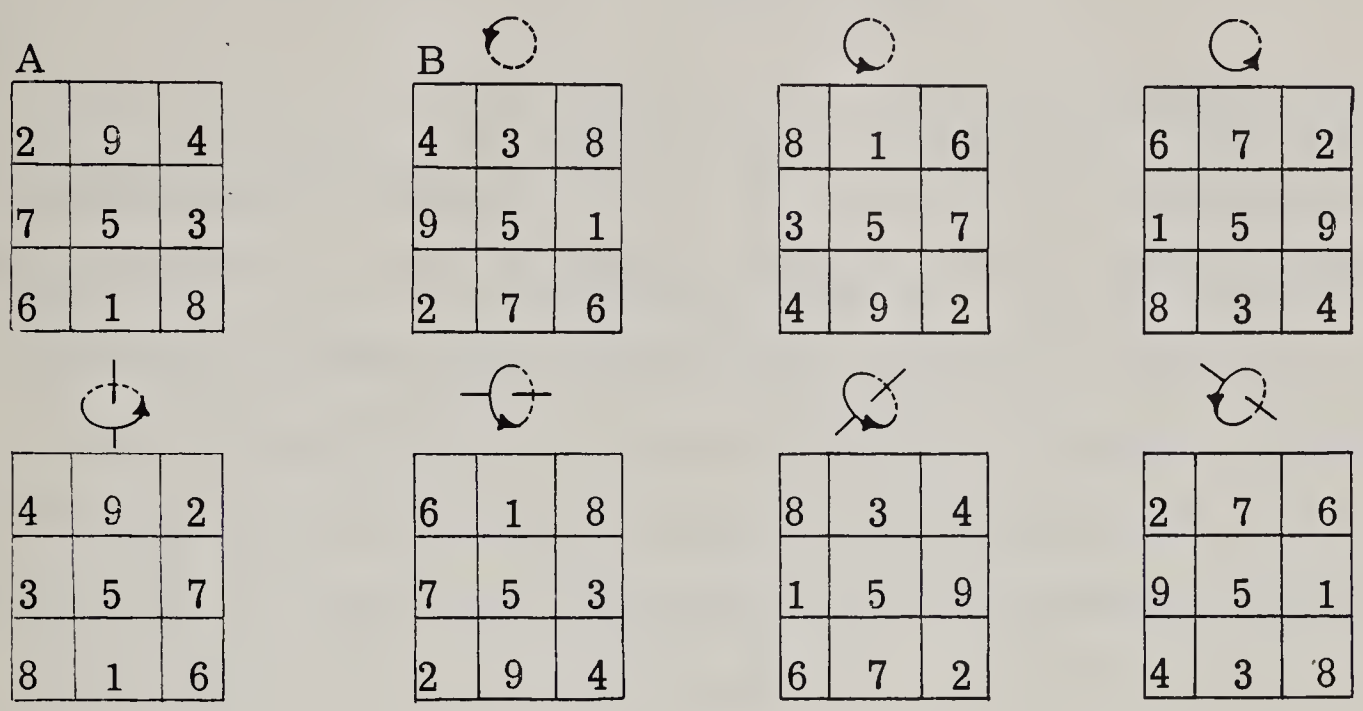
Pupils should be provided with Ditto sheets of scratch paper bearing blank Magic Square work forms with places to write the sums.

Another variation on the game of arranging the numbers 1-9 in a 3-by-3 square is to make the score the total of the sums for the rows, columns, and diagonals. For example, under this scoring rule, the scores for the four completed squares shown above are 126, 118, 113, and 120, respectively. Challenge pupils to discover the lowest and highest possible scores. This should be easy to do after they have discovered that for each completed square, the total of the row-sums and column-sums must be 90 (this is, of course, twice the sum of the numbers from 1 through 9). This means that the score for a completed square is 90 plus the total of the diagonal-sums. So, to find the lowest possible score, ...

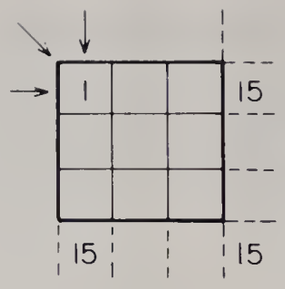
Exploring 3-by-3 Magic Squares

Notice that the completed squares in exercises A and B are magic "all the way."

The square in B can be obtained by rotating the square in A about its center a quarter-turn to the left. Six other Magic Squares can be obtained from the one in A by various rotations about the center or the center column or either diagonal.



It is natural to ask if these 8 completed squares are the only ones which are Magic Squares for the numbers 1-9. One could continue exploring but it is doubtful from a practical point of view that the answer would be obtained by exhaustion since there are 362, 880 different completed squares for the nine numbers from 1 through 9. (See Notes on Pupil Page 31.) In view of the very large number of possibilities it seems unlikely that merely 8 of them are Magic Squares. Yet, that is the case. As a way of seeing this, consider the possible positions for 1 in a completed square. Is there a Magic Square with 1 in the upper lefthand cell? If there were, there would have to be three



pairs of different numbers from $\{2, 3, 4, 5, 6, 7, 8, 9\}$ such that each pair together with 1 would have the sum 15. It turns out that there are only two such pairs, (5, 9) and (6, 8). So, there is no Magic Square with 1 in the upper left-hand cell. For the same reason, there is no Magic Square with 1 in any of the other corner cells. Nor is there a Magic Square with 1 in the center cell for that would require four triples of different numbers each with sum 15 and each having 1 as a component.

Therefore, the only position for 1 in a Magic Square is in the middle cell of an edge row or of an edge column. Let' s examine all the possibilities for 1 in the middle cell of the top row:

I

6	1	8
	5	
	9	

II

8	1	6
	5	
	9	

III

6	1	8
	9	
	5	

IV

8	1	6
	9	
	5	

V

5	1	9
	6	
	8	

VI

9	1	5
	6	
	8	

VII

5	1	9
	8	
	6	

VIII

9	1	5
	8	
	6	

As you can readily see, only two of these (I and II) can be Magic Squares, and these are the sixth and third of the Magic Squares shown in the earlier discussion. All other Magic Squares can be obtained from I (or II) by rotations.

Drill

Consider using as a drill sheet page 26 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Al has 14 bottle tops. Rudy has 7.
2. Jane figured she would weigh 18 pounds on the moon. People weigh 6 times as much on earth as they would on the moon.
3. Cloe' s teacher had 25 pieces of chalk. Each of eight children received a piece.

Headline for the Day:

Tell a story to fit this fact team:

$72 + 28 = \square$ $\qquad$ $\square - 28 = 72$

$28 + 72 = \square$ $\qquad$ $\square - 72 = 28$

NOTES ON PUPIL PAGE 31

Tasks: Pupils supply the missing sums and scores in accordance with the stated conditions on pages 30 and 31.

Purpose: To further explore Magic Squares while perfecting addition and subtraction skills.

Unifying Ideas: Addition and subtraction.

Materials for Teacher: Ditto masters.

Materials for Pupils: Ditto sheets of Magic Square forms.

The Lesson:

Numo might indeed get tired of arranging the numbers 1 through 9 before he had tried all possible arrangements. How many possible arrangements are there? There are 9 numbers which could be used to fill the first cell (top row, left). When the first cell has been filled, there are 8 numbers which could be used to fill the second cell (top row, middle). So, there are 9×8 pairs of numbers to fill the first two cells. (See Notes on Pupil Page 12.) When the first two cells have been filled, there are 7 numbers which could be used to fill the third cell. So, there are $9 \times 8 \times 7$ ways (that is, triples of numbers) to fill the first three cells. Continuing with this argument, it is easy to see that there are $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways to complete a 3-by-3 square with the numbers from 1 through 9.

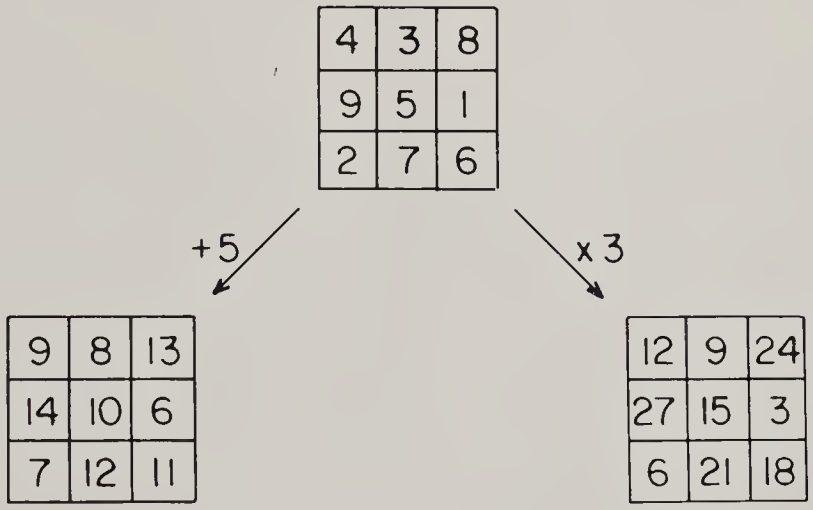
Exercises I and J explore the completed squares formed by using the first nine consecutive odd numbers. The "magic sum" for these nine numbers is 27. It is easy to find what the magic sum must be for any given numbers. Here's how: The total of the row-sums must be the sum of the given nine numbers. So, if the row-sums are to be equal, each row-sum must be one third of the sum of

the given numbers. In the case of the first nine consecutive odd numbers, their sum is 81; so, the magic sum is 27.

In similar fashion, we can compute the magic sum for the nine consecutive even numbers beginning with 2 (exercise K). The sum of these numbers is 90; so, the magic sum is 30.

More Exploring of 3-by-3 Magic Squares

As we mentioned in Notes on Pupil Page 30, there are exactly 8 Magic Squares with the numbers 1-9, and each of these can be obtained from any of the others by a rotation. One can also obtain Magic Squares from any of these by making certain kinds of systematic changes in all nine numbers. For example, consider the two completed squares obtained by adding 5 to each of the nine numbers in the square in B (on pupil page 30) and by tripling each of the numbers:



Both are Magic Squares. If you double each of the numbers in the square in B, you get the square in K. If you subtract 1 from each of the numbers in the square in K and rotate the resulting square a half-turn about its center row, you get the square in J.

Drill

Consider using as a drill sheet on time reckoning page 29 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Wally had 14 pencils. His brother gave him 12 more.
2. Gene is 8 years old. There are 12 months in 1 year.
3. Mila's third grade class bought 6 yards of wood to make puppets for their show. They found they needed 2 feet of wood to make each puppet.

Headline for the Day:

Fill in the blanks to complete the pattern.

600, 585, 570, , , , , , 480

NOTES ON PUPIL PAGE 32.

Tasks: For the exercises on the left half of the page, pupils record in the blanks either the number of different coins needed or the number of each kind of coin needed. In exercises A-D, pupils record the fewest coins for each amount of money specified in the tinted boxes and the fewest coins for the sum of the amounts specified in the tinted boxes.

Purpose: To explore numeration systems via the vehicle of the Fewest Coins game while simultaneously practicing computational skills.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Coins.

Materials for Pupils: Cuisenaire rods.

The Lesson:

Fewest Coins is a game we have played frequently in MATH WORKSHOP. On pupil pages 32 and 33 pupils are given practice in making amounts of money with the fewest possible coins. "Fewest coins" are then added to "fewest coins." The purpose of the activity is to provide experience with the idea of place value (or positional notation) in a new context--one in which the symbol "1 2 4" by temporary agreement stands not for one hundred twenty-four, but rather for 1 dime, 2 nickels, 4 pennies, or 24 cents.

A way of developing readiness for this page is to play the following game: I have coins worth 18¢ in my hand. What can you say for sure? Continue for several minutes. Then say: Now, I am going to give you some more information about the coins in my hand. The combination uses the fewest coins to make 18¢. Now, what can you say for sure? This new information abruptly halts the specu-

lation, because it has fully determined that the combination is 1 dime, 1 nickel, and 3 pennies. (Pupils can use Cuisenaire rods as they work toward the answer--the orange rods represent dimes, the yellow rods represent nickels, and the white rods represent pennies.)

Try the game again with another combination: What can you say for sure? Many things. And, finally, introduce the condition that it is a "fewest coins" combination. Again, the speculation is halted.

Ask: Is there any amount less than fifty cents that can be made up of the fewest coins in more than one way? Try several examples. (There are no such amounts!) The time spent in a fruitless search for such combinations is time well spent.

What Can You Say?

Try another kind of game: I gave the clerk a quarter. He gave me two coins as change. What can you say for sure about this situation? Answers may be:

If he gave you 2 dimes then you spent 5¢.

If he gave you 1 dime and 1 penny then you spent 14¢. Now ask the question: How many different amounts might I have spent? Make a chart of the numbers of coins he gave me:

D	N	P	Value	Spent
2			20	5
	2		10	15
		2	2	23
1	1		15	10
	1	1	6	19
1		1	11	14

Are all of these possibilities examples of making change with the "fewest coins"? (No--2 nickels is not; 1 dime would suffice.)

Let's make a chart for the fewest coins we need to make all amounts of 1¢ through 15¢.

Amounts	D	N	P
1¢	0	0	1
2¢	0	0	2
3¢	0	0	3
4¢	0	0	4
5¢	0	1	0
6¢	0	1	1

etc.

Drill

Consider using as a drill sheet page 30 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C.

Here are some stories for headline practice:

1. Samantha's sister is 21 years old. She is 13 years older than Samantha.
2. Cliff said he had 5 more minutes for lunch. There are 60 seconds in 1 minute.

Headline for the Day:

Tell a story to fit this fact team:

$185 + 15 = ____$; $____ - 15 = 185$

$15 + 185 = ____$; $____ - 185 = 15$

Date Due

CURR JUL 0'78			RETURN OCT 24'8
NOV 1 0 RETURN			
CURR APR 7 '84			
APR 7 RETURN			
FINAL APR 1 8'78			
MAR 3 0 RETURN			
JUL 2 3 RETURN			
JUL 2 3 RETURN			
EDUC E 1 5 '77			
FEB 1 1 RETURN			
DUE Educ MAY 31'78			
RETURNED MAY 23'78			
Date Due MAY 2 3 '79			
MAY 2 2 RETURN			
DUE Educ MAY 22'80			
MAY 22 RETURN			
DUE Educ OCT 18'83			

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